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Game-theoretic inductive definability

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Abstract

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We use game-theoretic ideas to define a generalization of the notion of inductive definability. This approach allows induction along non-well-founded trees. Our definition depends on an underlying partial ordering of the objects. In this ordering every countable ascending sequence is assumed to have a unique supremum which enables us to go over limits. We establish basic properties of this induction and examine examples where it emerges naturally. In the main results we prove an abstract Kleene Theorem and restricted versions of the Stage-Comparison Theorem and the Reduction Theorem.

1. Introduction

Let A be a set. An n-ary *inductive definition* on A is, according to [1], any mapping Γ from n-ary relations on A to n-ary relations on A which is monotone increasing, i.e., $R \subseteq S$ implies $\Gamma(R) \subseteq \Gamma(S)$. Every inductive definition has fixed points, i.e., relations R such that $\Gamma(R) = R$. The intersection of all these is again a fixed point called the *least* fixed point and denoted by Γ^∞. Suppose $\mathfrak{A}$ is a first-order structure. Every first-order formula $\phi(\boldsymbol{x}, S)$, where $\boldsymbol{x} = (x_1, \ldots, x_n)$ and S is an n-ary predicate symbol occurring positively in $\phi(\boldsymbol{x}, S)$, gives rise to a monotone increasing inductive definition as follows:

$$\Gamma_\phi(S) = \{\boldsymbol{a} \in A^n : \mathfrak{A} \models \phi(\boldsymbol{a}, S)\}.$$

According to [8], a relation $R(\boldsymbol{x})$ on $\mathfrak{A}$ is *inductively definable* on $\mathfrak{A}$, if there is a first-order formula $\phi(\boldsymbol{x}, \boldsymbol{y}, S)$ with S positive, and a sequence $\boldsymbol{b}$ of elements of A so that for all $\boldsymbol{a}$ in A: $R(\boldsymbol{a}) \Leftrightarrow \Gamma_\phi^\infty(\boldsymbol{a}, \boldsymbol{b})$.

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The concept of inductive definability is of fundamental importance throughout mathematics. The monograph [8] shows that although this concept was originally defined in the context of arithmetic, it can be defined on arbitrary structures and gives rise to a nice theory of its own. Maybe the most interesting application of inductive definability is the result (the so-called Kleene Theorem) that on a countable acceptable structure the class of inductive definable relations coincides with the class of relations which are Π^1_1-definable with parameters. This characterization is known to fail on uncountable structures. Applications of inductive definitions that are relevant from our point of view are: the (almost trivial) analysis of well-ordered sets, Cantor–Bendixson rank, analysis of the Ehrenfeucht–Fraïssé game or partial isomorphisms between structures, and syntax and semantics of infinitary languages $L_{\kappa\lambda}$.

Our purpose in this paper is to generalize the classical concept of inductive definability in a way which achieves the following two goals:

• The theory covers new areas, such as $\boldsymbol{\Pi}^1_1$-definability on uncountable structures, linear orderings with no descending α-sequences ($\alpha > \omega$), trees with no uncountable branches, Ehrenfeucht–Fraïssé games of length $>\omega$, and syntax and semantics of the extensions of $L_{\kappa\lambda}$ introduced in [13] and studied in [4, 9].

• A satisfactory general theory can be maintained.

To see how our generalization is defined, let us go back to some details of the classical notion. The standard construction of the least fixed point of an inductive definition is based on taking successive iterations:

$$R^0 = \emptyset, \qquad R^{\alpha+1} = \Gamma(R^\alpha), \qquad R^\nu = \bigcup_{\alpha<\nu} R^\alpha. \tag{1}$$

Now $R = \bigcup_\alpha R^\alpha$ is the least fixed point of Γ. An alternative definition is given in [1]. Aczel's characterization is game-theoretic. The following game has two players $\forall$ and $\exists$ (see Fig. 0). The rules of the game are that each player has to obey the condition displayed in Fig. 0. If he cannot move legally, the opponent

$\forall$	$\exists$	conditions
	A_0	$x \in \Gamma(A_0)$
x_1		$x_1 \in A_0$
	A_1	$x_1 \in \Gamma(A_1)$
x_2		$x_2 \in A_1$
	A_2	$x_2 \in \Gamma(A_2)$
⋮	⋮	⋮

Fig. 0. Aczel's game.

has won. Moreover, for $\exists$ to win, he has to win after a finite number of moves. Now the above least fixed point R satisfies $a \in R$ if and only if $\exists$ has a winning strategy in this game. The game can be easily formulated for sequences of elements instead of just elements.

Our generalized concept of inductive definability is based on Aczel's game. We allow this game to go on for up to ω_1 moves. Let us think for a moment what happens in Aczel's game if $\exists$ has not won the game during the first ω moves. The idea is that we form in a unique way a *limit* a_ω of the sequence $a_1, \ldots, a_i, \ldots$ and require $\exists$ to produce a set A_ω so that $a_\omega \in \Gamma(A_\omega)$. Now $\forall$ picks $a_{\omega+1} \in A_\omega$, and the game continues as before. But what is a_ω? We simply assume that there is an underlying partial ordering $\leqslant$ with the closure property that every countable ascending chain has a unique supremum. Additionally we demand that $\forall$ plays his a_i so that they form a $\leqslant$-ascending chain.

We get a generalized 'fixed point' construction by taking the set of a for which $\exists$ has a winning strategy in the game of length ω_1 described above. This leads naturally to a generalization of the notion of inductive definability on a structure.

In the traditional theory of inductive definability ordinals are used to denote stages of induction, such as Γ^α above. In our generalized framework this is not possible. Instead of ordinals we use *trees.* Ordinals present themselves in our approach as trees with no infinite branches, whereas we really allow all trees with no *uncountable* branches. With such trees we get a coherent theory of stages of induction with a Stage-Comparison Theorem.

It is a general feature of our theory that it is by far not as beautiful as the classical theory presented in [8] and many results have an element of incompleteness in them. This should come as no surprise, since we are after all dealing with 'non-well-founded' induction. An indication of the kind of difficulties that arise, note that while the Aczel game of length ω is determined as an open game, the corresponding game of length ω_1 need not be determined.

The structure of the paper is as follows. Section 2 describes some fundamental examples which have been the motivation behind the general theory. Section 3 gives some necessary preliminaries about trees. The tree-concept is fundamental in our study of induction. Section 4 gives the basic definitions of T-closure and T-coclosure of a monotone operator, as well as some examples. Section 5 discusses some variations of the basic definitions. In Section 6 we use the concepts of T-closure and T-coclosure to define the concepts of T-inductive and T-coinductive definability on a first order structure. In Section 7 we prove an Abstract Kleene Theorem which establishes a connection between ω_1-coinductive definability and Σ_1^1-definability on structures of cardinality ω_1 with enough coding. Section 8 introduces the concept of a stage of induction. These stages are trees with possible infinite branches but with no uncountable branches. In Section 9 we prove a Stage-Comparison Theorem. Finally, in Section 10 we use the Stage-Comparison Theorem to prove a restricted version of the Reduction Theorem.

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2. Preliminary examples

We shall discuss in this section some examples to indicate what kind of generalization of induction we want to cover with our general concepts.

2.1. Example. Consider the class of all linear orderings. Especially, $(A, \leqslant)$ will be always a linear ordering in this example. Our starting point is the observation that $(A, \leqslant)$ is a well-ordering, if and only if

(i) $A = \emptyset$, or

(ii) A has a greatest element a and $(A - \{a\}, \leqslant)$ is a well-ordering, or

(iii) there is a family $\mathscr{B}$ of proper initial segments of $(A, \leqslant)$ where $A = \bigcup \mathscr{B}$ and $(B, \leqslant)$ is a well-ordering for all $B \in \mathscr{B}$.

One can characterize the notion of a well-ordering on the basis of this observation in two different ways. Define first an operator Γ mapping sets of linear orderings to sets of linear orderings so that $(A, \leqslant) \in \Gamma(\mathscr{C})$, if and only if (i′) through (iii′) hold, where (i′) through (iii′) are obtained from (i) through (iii) by replacing 'is a well-ordering' by 'is in $\mathscr{C}$'. This operator Γ is monotone and its smallest fixed point is $\Gamma^{\infty} = \bigcup_{\alpha \in \mathrm{On}} \Gamma^{\alpha}$, where as usual, $\Gamma^0 = \emptyset$, $\Gamma^{\alpha+1} = \Gamma(\Gamma^{\alpha})$ and $\Gamma^{\alpha} = \bigcup_{\nu<\alpha} \Gamma^{\nu}$ for limit ordinals α. Of course, $(A, \leqslant)$ is in Γ^{∞}, if and only if it is in Γ^{κ} for $\kappa = \mathrm{card}(A)^+$. It is easy to see that Γ^{∞} is the class of all well-orderings.

Another way to use (i) through (iii) is to consider the following game $G((A, \leqslant), \omega)$ with players $\forall$ and $\exists$ where the players produce a descending sequence of initial segments of $(A, \leqslant)$ as follows: On the first round of the game $\forall$ plays $A_0 = A$. Then $\exists$ plays any collection $\mathscr{B}_0$ of initial segments of A_0 where $(A, \leqslant) \in \Gamma(\mathscr{B}_0)$. If $\mathscr{B}_0$ is empty, then $\exists$ has already won. Otherwise $\forall$ begins the next round by choosing some $A_1 \in \mathscr{B}_0$. Then $\exists$ has to give some $\mathscr{B}_1$ with $(A, \leqslant) \in \Gamma(\mathscr{B}_1)$. If $\mathscr{B}_1$ is empty, then $\exists$ has won. Otherwise, the players go to the next round. This is repeated ω times. Player $\exists$ wins the game, if he wins on some round $n < \omega$, i.e., if $\mathscr{B}_n$ is empty for some n. In this case $\forall$ loses. In case $\mathscr{B}_n \neq \emptyset$, for all n, neither of the players wins or loses. This kind of terminology concerning winning or losing will be used below in connection with other games, too. According to it, a player wins when the opponent cannot move.

The game defined above is our first example of what was called *Aczel's game* in the Introduction. Such games were first introduced in [1].

It is easy to see that $(A, \leqslant)$ is a well-ordering, if and only if player $\exists$ has a winning strategy in $G((A, \leqslant), \omega)$. Indeed, if $(A, \leqslant)$ does not contain infinite descending sequences, then no play of $G((A, \leqslant), \omega)$ can be infinite since $A_n - A_{n+1}$ is always nonempty if A_n is. On the other hand, if $a_0 > a_1 > \cdots$, then $\forall$ has an easy no-losing strategy: $\forall$ plays A_n always so that there is some m_n with $a_m \in A_n$ for all $m \geqslant m_n$.

Consider then a play of $G((A, \leqslant), \omega)$ where $\forall$ has not lost. Such a play determines a descending sequence $A = A_0 \supset A_1 \supset \cdots$ of initial segments of A. It is a natural question to ask what happens if we extend the game $G((A, \leqslant), \omega)$ so that the players can go on playing with the initial segment $A_\omega = \bigcap_{n<\omega} A_n$. This means that on round ω $\forall$ first plays A_ω and then $\exists$ has to play some $\mathscr{B}_\omega$ with $A_\omega \in \Gamma(\mathscr{B}_\omega)$. After this the players go on as in $G((A, \leqslant), \omega)$ with the addition that all limit steps in the game are passed by means of forming intersections like A_ω above. In $G((A, \leqslant), \omega_1)$ we let the players play in this way round α for all $\alpha < \omega_1$. Player $\exists$ wins and $\forall$ loses, if $\mathscr{B}_\alpha$ is empty for some $\alpha < \omega_1$.

In this case it is easy to see that $(A, \leqslant)$ does not contain descending ω_1-sequences, if and only if player $\exists$ has a winning strategy in $G((A, \leqslant), \omega_1)$ (if and only if $\forall$ does not have a no-losing strategy). Indeed, the argument sketched in connection with $G((A, \leqslant), \omega)$ works here.

We can conclude that the operator Γ can be used to define the notion of a linear orderings which does not have descending κ-sequences, when Γ is approached in terms of $G((A, \leqslant), \kappa)$ and κ is ω or ω_1. (This holds of course for other κ, too.) So the game-theoretic approach seems to be more versatile than the usual one based on the iterations of Γ. This is elaborated in [10].

2.2. Example. Consider an open game formula

$$\Phi = \forall x_0 \exists y_0 \forall x_1 \exists y_1 \cdots \bigvee_{n \leqslant \omega} \phi_n(x_0, \ldots, y_n)$$

where ϕ_n is first order. Its satisfaction in a structure $\mathfrak{A}$ is defined in terms of an obvious semantic game denoted by $G(\Phi, \mathfrak{A})$. In this game, the players produce a sequence $a_0, b_0, a_1, b_1, \ldots$ and $\exists$ wins, if this sequence makes some ϕ_n true. As is shown in [8], this kind of game sentences are closely connected to inductive definability. Consider the following operator Γ mapping sets of finite sequences of elements of A to sets of finite sequences of elements of A. We put $(a_0, b_0, \ldots, a_n, b_n)$ in $\Gamma(B)$, if and only if

$$\mathfrak{A} \models \bigvee_{k \leqslant n} \phi_n(a_0, b_0, \ldots, a_n, b_n), \quad \text{or}$$

$$\forall x_{n+1} \in A\, \exists y_{n+1} \in A\, [(a_0, b_0, \ldots, a_n, b_n, x_{n+1}, y_{n+1}) \in B].$$

It is easy to see that Γ is monotone. Denote the empty sequence by $\emptyset$.

Claim. *$\emptyset \in \Gamma^\infty$, if and only if $\mathfrak{A} \models \Phi$.*

Assume $\emptyset \in \Gamma^\infty$. We describe a winning strategy for $\exists$ in $G(\Phi, \mathfrak{A})$. Let α_0 be the smallest ordinal α with $\emptyset \in \Gamma^\alpha$. Then α_0 is of the form $\alpha' + 1$. Let $\forall$ play in $G(\Phi, \mathfrak{A})\, x_0$. By the definition of Γ there is some y_0 with $(x_0, y_0) \in \Gamma^{\alpha'}$. Fix such a y_0. Let α_1 be the smallest ordinal α with $(x_0, y_0) \in \Gamma^\alpha$. Then $\alpha_1 < \alpha_0$ and α_1 is of the form $\alpha'+1$. Again, by the definition of Γ, there is some y_1 with $(x_0, y_0, x_1, y_1) \in \Gamma^{\alpha'}$. Fix y_1. Let α_2 be the smallest ordinal α with $(x_0, y_0, z_1, y_1) \in \Gamma^{\alpha}$. Then $\alpha_2 < \alpha_1$ and α_2 is of the form $\alpha'+1$. This process is repeated as long as possible. Since it generates a descending sequence of ordinals $\alpha_0 > \alpha_1 > \alpha_2 > \cdots$, there has to be some n with $\alpha_{n+1} = 0$. This means that $(x_0, \ldots, y_0) \in \Gamma(\phi)$, and hence

$$\mathfrak{A} \models \bigvee_{k \leq n} \phi_k(x_0, \ldots, y_k).$$

So $\exists$ has won.

Assume then that $\mathfrak{A} \models \Phi$. Therefore $\exists$ has a winning strategy S in the game $G(\Phi, \mathfrak{A})$. We form a tree T as follows. Its root is the empty sequence $\emptyset$. The immediate successors of the root are all the sequences (x_0, y_0) where $\exists$ has used S to play y_0. More generally, if $t = (x_0, y_0, \ldots, x_n, y_n) \in T$ and not $\mathfrak{A} \models \bigvee_{k \leq n} \phi(x_0, \ldots, y_k)$, then the immediate successors of t are the sequences $(x_0, y_0, \ldots, x_n, y_n, x_{n+1}, y_{n+1})$ where $\exists$ has played y_{n+1} according to S. Since S is a winning strategy, T has only finite branches. We label T with ordinals so that for all nodes $t \in T$, the label $l(t)$ is the supremum of all the ordinals $l(t')+1$ where t' is an immediate successor of t in T. It is easy to verify by induction on an ordinal α that whenever $t = (x_0, y_0, \ldots, x_n, y_n)$ and $l(t) = \alpha$, then $t \in \Gamma^{\alpha+1}$. Hence especially, $\emptyset \in \Gamma^{\infty}$. This completes the proof of the claim.

As in Example 2.1, the operator Γ corresponds to a game $G(\emptyset, \omega)$ in the following way. First $\forall$ plays $t_0 = \emptyset$. Then $\exists$ plays some set B_0 with $t_0 \in \Gamma(B_0)$. Notice that t_0 is an intial segment of every element of B_0. After this $\forall$ plays some $t_1 \in B_0$ and $\exists$ plays some B_1 with $t_1 \in \Gamma(B_1)$, and so on. Player $\exists$ wins, if B_n is empty for some $n < \omega$. It is obvious that $G(\emptyset, \omega)$ and $G(\Phi, \mathfrak{A})$ are essentially the same game. Especially, $\exists$ has a winning strategy in one, if and only if $\exists$ has a winning strategy in the other, and $\forall$ has a no-losing strategy in one, if and only if $\forall$ has a no-losing strategy in the other. Actually, this observation could be used to give a different proof for the assertion above, since one can show directly that $\emptyset \in \Gamma^{\infty}$ is equivalent to $\exists$ having a winning strategy in $G(\emptyset, \omega)$.

Next we consider an open game sentence where the prefix has length ω_1. Let

$$\Psi = \forall x_0 \exists y_0 \cdots \forall x_\alpha \exists y_\alpha \cdots \bigvee_{\alpha < \omega_1} \phi_\alpha(x_0, \ldots, y_\alpha),$$

where ϕ_α is a formula of $L_{\infty\omega_1}$. Satisfaction is defined again in terms of an obvious game $G(\Psi, \mathfrak{A})$. We can extend the monotone operator Γ in a natural way so that it maps sets of countable sequences of A to sets of countable sequences of A. In this case $(a_0, b_0, \ldots, a_\alpha, b_\alpha) \in \Gamma(B)$, if and only if

$$\mathfrak{A} \models \bigvee_{\nu \leq \alpha} \phi_n(a_0, b_0, \ldots, a_\nu, b_\nu), \quad \text{or}$$

$$\forall x_{\alpha+1} \in A\ \exists y_{\alpha+1} \in A\ [(a_0, b_0, \ldots, a_\alpha, b_\alpha, x_{\alpha+1}, y_{\alpha+1}) \in B].$$

The game $G(\emptyset, \omega_1)$ is as $G(\emptyset, \omega)$ above, but now limit steps are passed by means of considering the limit (i.e., union) of the sequences considered before the limit. Assume for example that the players have played t_n and B_n for all $n < \omega$ and that $\exists$ has not yet won. Then by the definition of Γ there must exist a sequence $t = (x_0, y_0, \ldots, x_n, y_n, \ldots)$ where $t_n = (x_0, y_0, \ldots, x_n, y_n)$ for all $n < \omega$. Then on round ω player $\forall$ plays $t_\omega = t$ and $\exists$ has to play some set B_ω with $t_\omega \in \Gamma(B_\omega)$. From this the game goes on as before, and all other limit steps are passed in the same way. Player $\exists$ wins if $B_\alpha = \emptyset$ for some $\alpha < \omega_1$.

Also in this case $G(\emptyset, \omega_1)$ is essentially the same game as $G(\Psi, \mathfrak{A})$. Hence a game-theoretic idea makes it possible to use the operator Γ to give meaning to the game sentence Ψ of length ω_1. If Ψ and Γ are as above and if we repeat the argument of the Claim of this example, then we see that $\emptyset \in \Gamma^\infty$, if and only if the initial segment

$$\forall x_0 \exists y_0 \forall x_1 \exists y_1 \cdots \bigvee_{k \leq \omega} \phi(x_0, \ldots, y_n)$$

of Ψ holds in $\mathfrak{A}$.

2.3. Example. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two structures of the same similarity type and let κ be cardinal. We assume for notational simplicity that the domains A and B of $\mathfrak{A}$ and $\mathfrak{B}$ are disjoint. The Ehrenfeucht–Fraïssé game $\mathrm{EF}(\mathfrak{A}, \mathfrak{B}, \kappa)$ has κ rounds. On round α player $\forall$ first picks an element x_α from one of the structures and then $\exists$ replies with an element y_α from the other. Thus a play of $\mathrm{EF}(\mathfrak{A}, \mathfrak{B}, \kappa)$ produces a sequence $(a_\alpha)_{\alpha<\kappa}$ of elements of A and a sequence $(b_\alpha)_{\alpha<\kappa}$ of elements of B, where for all α, $\{a_\alpha, b_\alpha\} = \{x_\alpha, y_\alpha\}$. Player $\forall$ wins, if the mapping $a_\alpha \to b_\alpha$ is not a partial isomorphism, i.e., the sequences do not satisfy the same atomic formulas. Otherwise, $\exists$ wins. (Here our terminology differs from that used elsewhere.) It is well known that $\mathfrak{A} \equiv_{\infty\omega} \mathfrak{B}$, if and only if player $\exists$ has a winning strategy in $\mathrm{EF}(\mathfrak{A}, \mathfrak{B}, \omega)$. Indeed, the latter condition is easily seen to be equivalent to the two structures being partially isomorphic, $\mathfrak{A} \equiv_{\mathrm{p}} \mathfrak{B}$. In a similar way $\mathrm{EF}(\mathfrak{A}, \mathfrak{B}, \kappa)$ characterizes elementary equivalence in a certain infinitely deep language, see [4]. Notice also that if $\mathfrak{A}$ and $\mathfrak{B}$ are of cardinality $\leq \kappa$, then either the two structures are isomorphic, and $\exists$ wins in $\mathrm{EF}(\mathfrak{A}, \mathfrak{B}, \kappa)$ by playing according to an isomorphism, or they are not isomorphic, and $\forall$ wins by going through all elements of the two structures. Thus $\mathrm{EF}(\mathfrak{A}, \mathfrak{B}, \kappa)$ is closed determined in this case in the sense that either $\exists$ has a no-losing strategy, or $\forall$ has a winning strategy. The Ehrenfeucht–Fraïssé games are further analyzed in [6] and [7].

We consider here the cases $\kappa = \omega$ and $\kappa = \omega_1$. Define an operator Γ mapping sets of countable sequences of $A \cup B$ to sets of countable sequences as follows. Denote first by N the set of those sequences $(x_0, y_0, \ldots, x_\nu, y_\nu, \ldots)_{\nu<\alpha}$ where $\exists$ has already lost, i.e., the corresponding sequences $(a_0, \ldots, a_\nu, \ldots)_{\nu<\alpha}$ and $(b_0, \ldots, b_\nu, \ldots)_{\nu<\alpha}$ do not satisfy the same atomic formulas. Then define

$(x_0, y_0, \ldots, x_\nu, y_\nu, \ldots)_{\nu<\alpha} \in \Gamma(C)$, if and only if

$$\begin{aligned}&(x_0, y_0, \ldots, x_\nu, y_\nu, \ldots)_{\nu<\alpha} \in N, \quad \text{or}\\ &\exists x_\alpha \in A \cup B\ \forall y_\alpha \in A \cup B\ [[x_\alpha \in A \Leftrightarrow y_\alpha \in B]\\ &\quad \Rightarrow (x_0, y_0, \ldots, x_\nu, y_\nu, \ldots)_{\nu\leq\alpha} \in C].\end{aligned}$$

It is easy to see that this operator inductively defines the notion of partial isomorphism in the following sense.

Claim 1. (i) $\forall$ *has a winning strategy in* $\mathrm{EF}(\mathfrak{A}, \mathfrak{B}, \omega)$, *if and only if* $\emptyset \in \Gamma^\infty$.
(ii) $\exists$ *has a winning strategy in* $\mathrm{EF}(\mathfrak{A}, \mathfrak{B}, \omega)$, *if and only if* $\emptyset \in \Gamma^\infty$.

Notice first that $\mathrm{EF}(\mathfrak{A}, \mathfrak{B}, \omega)$ is determined by the Gale–Stewart theorem. So (ii) follows from (i). Assertion (i) is proved very much like the Claim of the previous example. Assume that $\emptyset \in \Gamma^\infty$. Then $\emptyset \in \Gamma(\Gamma^\beta) - \Gamma^\beta$ for some β. In this case there is some x_0 such that for all y_0, $(x_0, y_0) \in \Gamma^\beta$. Then the first move of $\forall$ will be x_0 to which $\exists$ replies with some y_0. But there must be some $\nu < \beta$ with $(x_0, y_0) \in \Gamma(\Gamma^\nu) - \Gamma^\nu$. If x_0 and y_0 satisfy the same atomic formulas, i.e., $(x_0, y_0) \notin N$, then this argument will be repeated. Since it leads to a descending sequence of ordinals, there has to be some $n < \omega$ with $(x_0, y_0, \ldots, x_n, y_n) \in N$. In this case $\forall$ wins.

Assume then that $\forall$ has a winning strategy S in $\mathrm{EF}(\mathfrak{A}, \mathfrak{B}, \omega)$. Then as in the proof of the Claim of the previous example, we form a tree T consisting of such initial segments of a play of $\mathrm{EF}(\mathfrak{A}, \mathfrak{B}, \omega)$ where $\forall$ uses S and $\exists$ has not yet lost. Thus the empty sequence $\emptyset$ is the unique root of T. We label this tree with ordinals as before. If α is the label of $\emptyset$, then it is easy to show that $\emptyset \in \Gamma^{\alpha+1}$. This completes the proof of Claim 1.

The operator Γ corresponds again to a game $G(\Gamma, \emptyset, \omega)$. The first move of $\forall$ is to play $s_0 = \emptyset$. To this $\exists$ has to respond with a set C_0 where $s_0 \in \Gamma(C_0)$. This means that there has to be $x_0 \in A \cup B$ where for every $y_0 \in A \cup B$ taken from a different structure than x_0, it holds that $(x_0, y_0) \in C_0$. Then $\forall$ chooses some $s_1 \in C_0$ and $\exists$ has to play a set C_1 where $s_1 \in \Gamma(C_1)$, etc.

It is easy to see that the roles of $\exists$ and $\forall$ in $G(\Gamma, \emptyset, \omega)$ correspond to those of $\forall$ and $\exists$ in $\mathrm{EF}(\mathfrak{A}, \mathfrak{B}, \omega)$. Indeed, to play C_0, $\exists$ has essentially to choose at least one x_0 as above, and conversely. And to choose s_1, $\forall$ has essentially to choose y_0, and conversely. The following claim follows easily from this observation.

Claim 2. (i) *Player* $\forall$ *has a winning strategy in* $\mathrm{EF}(\mathfrak{A}, \mathfrak{B}, \omega)$, *if and only if player* $\exists$ *has a winning strategy in* $G(\Gamma, \emptyset, \omega)$.
(ii) *Player* $\exists$ *has a winning strategy in* $\mathrm{EF}(\mathfrak{A}, \mathfrak{B}, \omega)$, *if and only if player* $\forall$ *has a no-losing strategy in* $G(\Gamma, \emptyset, \omega)$.

The game $G(\Gamma, \emptyset, \omega)$ of this example corresponds closely to the analogous one in the previous example. Especially, player $\forall$ chooses in both games longer and longer sequences. Hence $G(\Gamma, \emptyset, \omega)$ can be easily extended to a game $G(\Gamma, \emptyset, \omega_1)$ of length ω_1 so that limit steps in the game are passed by means of forming the union (limit) of the sequences played by $\forall$ earlier in the game. The idea behind Claim 2 easily yields the following observation.

Claim 3. (i) *Player $\forall$ has a winning strategy in* EF$(\mathfrak{A}, \mathfrak{B}, \omega_1)$, *if and only if player $\exists$ has a winning strategy in $G(\Gamma, \emptyset, \omega_1)$.*

(ii) *Player $\exists$ has a winning strategy in* EF$(\mathfrak{A}, \mathfrak{B}, \omega_1)$, *if and only if player $\forall$ has a no-losing strategy in $G(\Gamma, \emptyset, \omega_1)$.*

So once more we are in a situation where an object EF$(\mathfrak{A}, \mathfrak{B}, \kappa)$ can be represented by means of Γ^∞ in the special case $\kappa = \omega$, but Γ is connected to EF$(\mathfrak{A}, \mathfrak{B}, \kappa)$ also in other cases via games related to Γ in a uniform way.

2.4. Example. Consider a closed game formula

$$\Phi = \forall x_0 \exists y_0 \cdots \bigwedge_{n<\omega} \phi_n(x_0, y_0, \ldots, x_n, y_n).$$

The satisfaction relation is defined by means of an obvious semantic game $G(\Phi, \mathfrak{A})$ of length ω. So Φ holds in $\mathfrak{A}$, if and only if player $\exists$ has a no-losing strategy in $G(\Phi, \mathfrak{A})$. This relation is closely connected to a monotone operator Γ mapping sets of finite sequences of the domain A to sets of finite sequences, and where $(x_0, y_0, \ldots, x_n, y_n) \in \Gamma(B)$, if and only if

$$\mathfrak{A} \nvDash \bigwedge_{k\leq n} \phi_k(x_0, y_0, \ldots, x_k, y_k), \quad \text{or}$$

$$\exists x_{n+1} \in A\ \forall y_{n+1} \in A\ [(x_0, y_0, \ldots, x_{n+1}, y_{n+1}) \in B].$$

The following assertion is easily proved by the arguments of the previous example. Actually the previous example can be seen to be a special case of the present one. Indeed, the Ehrenfeucht–Fraïssé game between two structures $\mathfrak{A}$ and $\mathfrak{B}$ can be presented in the form $G(\Phi, \mathfrak{A} + \mathfrak{B})$ where Φ is a suitable game sentence and $\mathfrak{A} + \mathfrak{B}$ is a suitable version of the disjoint union of $\mathfrak{A}$ and $\mathfrak{B}$.

Claim 1. *$\mathfrak{A} \vDash \Phi$, if and only if $\emptyset \in \Gamma^\infty$.*

We can define a game $G(\Gamma, \emptyset, \omega)$ related to Γ as in the previous examples. Then again by earlier arguments, we have the following sharper version of the previous observation.

Claim 2. (i) *Player $\exists$ has a no-losing strategy $G(\Phi, \mathfrak{A})$, if and only if player $\forall$ has a no-losing strategy in $G(\Gamma, \emptyset, \omega)$.*

(ii) *Player $\forall$ has a winning strategy in $G(\Phi, \mathfrak{A})$, if and only if player $\exists$ has a winning strategy in $G(\Gamma, \emptyset, \omega)$.*

Consider next an analogous game sentence with a prefix of length ω_1,

$$\Phi = \forall x_0 \exists y_0 \cdots \forall x_\nu \exists y_\nu \cdots \bigwedge_{\nu<\omega_1} \phi_\nu(x_0, y_0, \ldots, x_\nu, y_\nu).$$

Satisfaction is again defined in terms of an obvious semantic game, in this case of length ω_1. We extend the definition of the operator Γ to map sets of countable sequences of elements of A to sets of countable sequences of A by defining $(x_0, y_0, \ldots, x_\nu, y_\nu)_{\nu<\alpha} \in \Gamma(B)$, if and only if

$$\mathfrak{A} \nvDash \bigwedge_{\nu<\alpha} \phi_\nu(x_0, y_0, \ldots, x_\nu, y_\nu), \quad \text{or}$$

$$\exists x_\alpha \in A\, \forall y_\alpha \in A\, [(x_0, y_0, \ldots, x_\alpha, y_\alpha) \in B].$$

Also the game $G(\Gamma, \emptyset, \omega_1)$ is defined as before. Limit steps of this game are again passed by considering the union of the sequences played earlier by $\forall$. The following assertion follows again from the arguments of the previous example.

Claim 3. (i) *Player $\exists$ has a no-losing strategy in $G(\Phi, \mathfrak{A})$, i.e., $\mathfrak{A} \vDash \Phi$, if and only if player $\forall$ has a no-losing strategy in $G(\Gamma, \emptyset, \omega_1)$.*

(ii) *Player $\forall$ has a winning strategy in $G(\Phi, \mathfrak{A})$, if and only if player $\exists$ has a winning strategy in $G(\Gamma, \emptyset, \omega_1)$.*

Notice that as in Example 2.2, Γ^∞ depends only on the initial segment of (the prefix of) Φ of length ω.

3. Trees

We shall use trees rather than ordinals to measure stages of induction. For this end we review here some basic facts about trees. By a *tree* we mean any partial ordering $(T, \leqslant_T)$ in which the set of predecessors $\{t' : t' <_T t\}$ of every element $t \in T$ is well-ordered by $\leqslant_T$. We do not require that trees have a unique root. We shall assume, for convenience, that all trees have height $\leqslant \omega_1$.

A good example of a tree in this connection is the tree $T(A)$ of all ascending sequences, closed under supremum of subsets, of elements of a subset A of ω_1. If A is co-stationary, then $T(A)$ has no uncountable branches.

Any ordinal α is a tree as a linearly ordered set. We use simply α to denote the ordinal α as a linearly ordered tree. There is also another way of construing an ordinal as a tree: If α is an ordinal, we let B_α denote the tree of all non-empty descending chains of elements of α ordered as follows: $s \leqslant s'$ iff s is an initial segment of s'.

We shall need two different ordering relations between trees. We write $T \leqslant U$ if there is an order-preserving mapping from the tree T into the tree U. This mapping need not be one to one. If $T \leqslant U$ but not $U \leqslant T$, we write $T < U$. Finally if $T \leqslant U$ and $U \leqslant T$, we write $T \equiv U$. It is easy to see that $\leqslant$ is a partial ordering of the $\equiv$-equivalence classes of trees.

Our second ordering relation between trees is based on the following construction. Let σT be the tree of all countable initial segments of branches of the tree T. We define $T \ll U$ if $\sigma T \leqslant U$. Suppose T has no uncountable branches. Then it is easy to see that $T < \sigma T$. The following properties of $\ll$ are all easy to verify directly (they are proved in [6]):

(1) $T \ll U$ implies $T < U$.
(2) $T \ll \sigma T$.
(3) $\neg \exists U\,(T \ll U \ll \sigma T)$.
(4) $\ll$ is well-founded below T.

The main difference between $T \ll U$ and $T < U$ arises, roughly speaking, from the fact that the first asserts the existence of a mapping whereas the second asserts the lack of a mapping. If $A \subseteq B \subseteq \omega_1$ such that the sets A, $B - A$ and $\omega_1 - B$ are all stationary, then $T(A) < T(B)$ but not $T(A) \ll T(B)$. This is proved in [6]. Notice, that if T has an uncountable branch, then $T \equiv \sigma T$. If the word 'countable' is dropped from the definition of σT, then (1)–(4) above hold for all T, but σT may have height $= \omega_1 + 1$.

4. The T-closure of a monotone operator

In this section we generalize Aczel's game and define the notions of the T-closure and the T-coclosure of a monotone operator.

Let $\boldsymbol{X} = \langle X, \leqslant \rangle$ be a partially ordered structure in which every countable ascending sequence $(x_\alpha)_{\alpha<\nu}$ has a unique supremum $\lim_{\alpha<\nu} x_\alpha$. For example, $\boldsymbol{X}$ could be the set of all subsets of a domain with the ordering $x \leqslant y$ iff $y \subseteq x$. Or $\boldsymbol{X}$ could be the set of all sequences of a domain with the ordering $s \leqslant s'$ iff s is an initial segment of s'.

A *monotone operator* on $\boldsymbol{X}$ is a function $\Gamma : \mathcal{P}(X) \to \mathcal{P}(X)$ such that $A \subseteq B$ implies $\Gamma(A) \subseteq \Gamma(B)$ for any $A, B \subseteq X$. A set A is *Γ-dense*, if $A \subseteq \Gamma(A)$, *Γ-closed*, if $\Gamma(A) \subseteq A$, and a *fixed point of Γ*, if $A = \Gamma(A)$.

Let Γ be a monotone operator on $\boldsymbol{X}$, T a tree of height $\leqslant \omega_1$ and x an element of X. We shall consider the following two-person game, a modification of Aczel's game, $G(\Gamma, x, T)$: The players are $\exists$ and $\forall$ and there are at most ω_1 moves. Player $\exists$ moves first and always after a sequence of moves of limit ordinal length. He starts with A_0 such that $x \in \Gamma(A_0)$ and $y \geqslant x$ for all $y \in A_0$. Then $\forall$ moves $x_1 \geqslant x$ from A_0. Whenever $\forall$ has moved x_α, $\exists$ plays A_α with $x_\alpha \in \Gamma(A_\alpha)$ where $y \geqslant x_\alpha$ for all $y \in A_\alpha$, and then $\forall$ moves $x_{\alpha+1} \in A_\alpha$ with $x_{\alpha+1} \geqslant x_\alpha$. At limit stages $\exists$ moves A_ν such that $x_\nu = \lim_{\alpha<\nu} x_\alpha \in \Gamma(A_\nu)$ and then $\forall$ moves $x_{\nu+1} \in A_\nu$. At each move $\exists$ has to play also some element of T in such a way that these elements form an ascending chain in T. See Fig. 1 for a picture of the game. Player $\exists$ *loses* if he cannot play A_α or t_α. Then $\forall$ *wins*. Player $\forall$ *loses* if he cannot play $x_{\alpha+1}$. Then $\exists$ *wins*. We say that Γ is *open determined* on T if for all x $\exists$ has a winning

$\forall$	$\exists$	conditions (see the text for an exact definition)
	A_0, t_0	$x \in \Gamma(A_0)$
x_1		$x_1 \in A_0$, $x_1 \geq x$
	A_1, t_1	$x_1 \in \Gamma(A_1)$, $t_0 < t_1$
x_2		$x_2 \in A_1$, $x_2 \geq x_1$
	A_2, t_2	$x_2 \in \Gamma(A_2)$, $t_1 < t_2$
.	.	.
.	.	.
.	.	.
	A_ω, t_ω	$x_\omega \in \Gamma(A_\omega)$, $t_n < t_\omega$ all $n < \omega$ ($x_\omega = \lim_{n<\omega} x_n$)
$x_{\omega+1}$	.	.
.	.	.
.	.	.
.	.	.

Fig. 1. The game $G(\Gamma, x, T)$.

strategy or $\forall$ has a no-losing strategy in $G(\Gamma, x, T)$. Respectively, we say that Γ is *closed determined* on T if for all x $\exists$ has a no-losing strategy or $\forall$ has a winning strategy in $G(\Gamma, x, T)$. By the Gale–Stewart theorem every operator is both open and closed determined on a tree which has no branches of length $>\omega$.

Definition. The *T-closure* of a monotone operator Γ on X is the set

$$\mathrm{Ind}(\Gamma, T) = \{x \in X \mid \exists \text{ has a winning strategy in } G(\Gamma, x, T)\}.$$

The *T-coclosure* of Γ is the set

$$\mathrm{Coind}(\Gamma, T) = \{x \in X \mid \forall \text{ has a no-losing strategy in } G(\Gamma, x, T)\}.$$

Remark. Trivial properties of these sets are:

(1) $\mathrm{Ind}(\Gamma, T) \cap \mathrm{Coind}(\Gamma, T) = \emptyset$.

(2) If $T \leqslant T'$, then $\mathrm{Ind}(\Gamma, T) \subseteq \mathrm{Ind}(\Gamma, T')$ and $\mathrm{Coind}(\Gamma, T') \subseteq \mathrm{Coind}(\Gamma, T)$.

(3) If Γ is open determined on T, then $\mathrm{Ind}(\Gamma, T) \cup \mathrm{Coind}(\Gamma, T) = X$.

The first property means that $\mathrm{Ind}(\Gamma, T) \subseteq X\text{-Coind}(\Gamma, T)$. It will turn out that the difference $X\text{-Coind}(\Gamma, T)$ behaves in many situations like a version of 'the T-closure' of Γ. We denote $X\text{-Coind}(\Gamma, T)$ by $\mathrm{Ind}'(\Gamma, T)$. If Γ is open determined on T, then $\mathrm{Ind}(\Gamma, T) = \mathrm{Ind}'(\Gamma, T)$.

4.1. Example. It is easily proved that

$$\mathrm{Ind}(\Gamma, B_0) = \emptyset,$$

$$\mathrm{Ind}(\Gamma, B_{\alpha+1}) = \Gamma(\mathrm{Ind}(\Gamma, B_\alpha)), \quad \text{and}$$

$$\mathrm{Ind}(\Gamma, B_\nu) = \bigcup_{\alpha<\nu} \mathrm{Ind}(\Gamma, B_\alpha), \quad \text{for } \nu = \bigcup \nu,$$

that is, $\mathrm{Ind}(\Gamma, B_\alpha) = \Gamma^\alpha$. Let Γ^* be the *dual* of Γ, i.e., $\Gamma^*(A) = X - \Gamma(X - A)$. We denote $\mathrm{Coind}(\Gamma^*, T)$ by $\mathrm{Ker}(\Gamma, T)$, and call it the *T-kernel* of Γ. Clearly

$$\mathrm{Ker}(\Gamma, B_0) = X,$$

$$\mathrm{Ker}(\Gamma, B_{\alpha+1}) = \Gamma(\mathrm{Ker}(\Gamma, B_\alpha)), \quad \text{and}$$

$$\mathrm{Ker}(\Gamma, B_\nu) = \bigcap_{\alpha<\nu} \mathrm{Ker}(\Gamma, B_\alpha), \quad \text{for } \nu = \bigcup \nu.$$

Moreover, $\mathrm{Ind}(\Gamma, \omega)$ is the least fixed point of Γ and hence

$$\mathrm{Ind}(\Gamma, \omega) = \bigcup_{\alpha\in\mathrm{On}} \mathrm{Ind}(\Gamma, B_\alpha).$$

Also

$$\mathrm{Ker}(\Gamma, \omega) = \bigcap_{\alpha\in\mathrm{On}} \mathrm{Ker}(\Gamma, B_\alpha).$$

The former of the equations tells that $\mathrm{Ind}(\Gamma, \omega)$ is the set Γ^∞ *inductively defined* by Γ, and the latter equation means that $\mathrm{Ker}(\Gamma, \omega)$ is the *kernel* Γ_∞ of Γ.

4.2. Example. Let $\boldsymbol{X}$ be the class of all trees ordered by $T \leqslant T'$ iff T' is a subtree of T. We define the following sum-operation in $\boldsymbol{X}$. Let A be a set of trees. Let S be the union of A and the set of pairs (t, T), where $t \in T \in A$. Let $\leqslant$ be the partial ordering of S determined by the conditions:

$$T \leqslant (t, T), \quad \text{if } t \in T \in A,$$

$$(t, T) \leqslant (t', T), \quad \text{if } t, t' \in T \in A \text{ and } t \leqslant t' \text{ in } T.$$

We call S the *sum* of the trees in A. Note that B_α is equivalent to the sum of the trees B_β, $\beta < \alpha$. The sum-operation gives rise to the following monotone operator on $\boldsymbol{X}$. For $A \subseteq \boldsymbol{X}$:

$$T \in \Gamma_s(A) \Leftrightarrow T \text{ is either empty or a sum of trees in } A.$$

4.3. Lemma. *The following are equivalent for any tree T and any tree U with no uncountable branches*:

(1) $U \ll T$,

(2) $U \in \mathrm{Ind}(\Gamma_s, T)$.

Proof. (1)$\Rightarrow$(2). Let f be an order-preserving mapping $\sigma U \to T$. Player $\exists$ wins $G(\Gamma_s, U, T)$ as follows: He starts with

$$A_0 = \{\{t \in U \mid t > a\} \mid a \text{ is a minimal element of } U\}$$

and $t_0 = f(\emptyset)$. Whenever $\forall$ has played $x_{\alpha+1}$, $\exists$ plays

$$A_{\alpha+1} = \{\{t \in x_{\alpha+1} \mid t > a\} \mid a \text{ is a minimal element of } x_{\alpha+1}\}.$$

Considering that $x_{\alpha+1} = \{t \in x_\alpha \mid t > a\}$ for some $a \in x_\alpha$, $\exists$ can let $t_{\alpha+1}$ be $f(\{b \in U \mid b \leqslant a\})$. At limits $\exists$ takes the intersection x_ν of the trees played by $\forall$ and defines A_ν as above. The move t_ν is the value of f at the branch of U determined by the previous moves in the game. This is a winning strategy for $\exists$ since he cannot lose and the game cannot go on for uncountably many moves.

(2)$\Rightarrow$(1). Let τ be a winning strategy of $\exists$ in $G(\Gamma_s, U, T)$. the order-preserving mapping $f : \sigma U \to T$ is defined as follows. Suppose b is a branch in U. This branch determines a partial strategy ρ of $\forall$ in $G(\Gamma_s, U, T)$. Let us play $G(\Gamma_s, U, T)$ as long as $\forall$ can follow ρ, $\exists$ playing τ. After this partial play τ gives $\exists$ a new move $t_\alpha \in T$. We let $f(b) = t_\alpha$. $\square$

4.4. Corollary. *If U is a tree with no uncountable branches, then*

$$U \in \mathrm{Ind}(\Gamma_s, \sigma U) - \mathrm{Ind}(\Gamma_S, U).$$

4.5. Lemma. *The following are equivalent for any trees T and U*:

(1) $T \leqslant U$,

(2) $U \in \mathrm{Coind}(\Gamma_s, T)$.

Proof. (1)$\Rightarrow$(2). Player $\forall$ can avoid losing $G(\Gamma_s, U, T)$ by using $T \leqslant U$ to transfer moves of $\exists$ in T to his own moves among subtrees of U.

(2)$\Rightarrow$(1). Suppose $\forall$ has a no-losing strategy τ in $G(\Gamma_s, U, T)$. Let $t \in T$. We can let $\exists$ play the branch $\{t' \mid t' \leqslant t\}$ in $G(\Gamma_s, U, T)$ while $\forall$ follows τ. This yields an element $f(t)$ of U. Now the mapping f is order-preserving. $\square$

4.6. Corollary. *If U is any tree, then* $U \in \mathrm{Coind}(\Gamma_s, U) - \mathrm{Coind}(\Gamma_s, \sigma U)$.

4.7. Example. Let $A \subseteq \{0, 1\}^\omega$ be non-determined. Let $X = \{0, 1\}^{<\omega} \cup \{0, 1\}^\omega$ with the ordering $t \leqslant t'$ iff t is an initial segment of t'. Let

$$f \in \Gamma(B) \Leftrightarrow [\mathrm{dom}(f) \text{ finite and } \forall a\, \exists b\, (f \cup \{(a, b)\} \in B]$$
$$\vee\, [\mathrm{dom}(f) = \omega \text{ and } f \in A].$$

Neither $\exists$ nor $\forall$ has a no-losing strategy in $G(\Gamma, f, T)$ for $f \in X$ and for T with a branch of length $\geqslant \omega + 1$. Thus for such T, $\emptyset \notin \operatorname{Ind}(\Gamma, T) \cup \operatorname{Coind}(\Gamma, T)$.

4.8. Example. Let X be the set of countable sequences of elements of ω_1 with the ordering $x \leqslant y$ iff x is an initial segment of y. A subset x of ω_1 is said to be *closed* if it is closed under supremums of ascending sequences of its elements. We use CUB to denote the set of closed and unbounded subsets of ω_1. Let

$$x \in \Gamma_A(B) \Leftrightarrow B \text{ contains all proper extensions of } x \text{ to a closed sequence of elements of } A.$$

It is easy to see that

$$\emptyset \in \operatorname{Ind}(\Gamma_A, T) \Leftrightarrow T(A) \ll T,$$

$$\emptyset \in \operatorname{Coind}(\Gamma_A, T) \Leftrightarrow T \leqslant T(A).$$

In the first equivalence the idea is the following. Suppose $\emptyset \in \operatorname{Ind}(\Gamma_A, T)$ and $s \in \sigma(T(A))$. We let $\forall$ play elements of s successively in $G(\Gamma_A, \emptyset, T)$. When s ends, $\exists$ can still play an element $f(s)$ of T. The function f demonstrates $\sigma(T(A)) \leqslant T$. On the other hand, if f maps $\sigma T(A)$ order-preservingly to T, $\exists$ can play $G(\Gamma_A, \emptyset, T)$ as follows. If he has to find B so that $x \in \Gamma(B)$, he lets

$$B = \{y \in T(A): x \text{ is a proper initial segment of } y\}.$$

Then $x \in \Gamma_A(B)$. His move in T he gets with f from the moves of $\forall$. The second equivalence is proved similarly.

Notice also that

$$T(A) \ll \omega_1 \Leftrightarrow A \notin CUB,$$

$$\omega_1 \leqslant T(A) \Leftrightarrow A \in CUB.$$

Hence we have

$$\emptyset \in \operatorname{Ind}(\Gamma_A, \omega_1) \Leftrightarrow A \notin CUB,$$

$$\emptyset \in \operatorname{Coind}(\Gamma_A, \omega_1) \Leftrightarrow A \in CUB,$$

and

$$\emptyset \in \operatorname{Ind}(\Gamma_A, \omega) \Leftrightarrow A \text{ is finite},$$

$$\emptyset \in \operatorname{Coind}(\Gamma_A, \omega) \Leftrightarrow A \text{ is infinite}.$$

Let A and B be two disjoint stationary sets. Then $T(A) \not\leqslant T(B)$ and $T(B) \not\leqslant T(A)$ (see [6]). So

$$\emptyset \notin \operatorname{Ind}(\Gamma_A, T(B)) \cup \operatorname{Coind}(\Gamma_A, T(B)).$$

4.9. Example. Let $\mathfrak{A}$ and $\mathfrak{B}$ be models of the same language. Let X be the set of countable sequences s of pairs (a, b) where $a \in A$ and $b \in B$, ordered by end-extension. Let X_0 be the set of sequences which are partial isomorphisms between $\mathfrak{A}$ and $\mathfrak{B}$. We define a monotone operator Γ_{ef} on X as follows:

$$s \in \Gamma_{\mathrm{ef}}(C) \Leftrightarrow \exists a \,\forall b\, [s^\frown(a, b) \in X_0 \Rightarrow s^\frown(a, b) \in C] \\ \vee\, \exists b\, \forall a\, [s^\frown(a, b) \in X_0 \Rightarrow s^\frown(a, b) \in C].$$

Clearly, $\emptyset$ is in the ω-coclosure of Γ_{ef} if and only if $\mathfrak{A}$ and $\mathfrak{B}$ are partially isomorphic. Similarly, $\emptyset$ is in the ω_1-coclosure of Γ_{ef} if and only if the second player has a winning strategy in the Ehrenfeucht–Fraïssé game of length ω_1 between $\mathfrak{A}$ and $\mathfrak{B}$. There is an infinitary language $M_{\infty\omega_1}$ with the property that $\emptyset$ is in the ω_1-coclosure of Γ_{ef} if and only if $\mathfrak{A}$ and $\mathfrak{B}$ are equivalent relative to $M_{\infty\omega_1}$. For details concerning $M_{\infty\omega_1}$ we refer to [4] and [9]. In a sense, $M_{\infty\omega_1}$ has similar relation to $L_{\infty\omega}$ as the ω_1-closure of a monotone operator has to its ω-closure. The relation $T \in \mathrm{Coind}(\Gamma_{\mathrm{ef}}, T)$ is studied in detail in [6].

4.10. Example. A well-known monotone operator on a topological space E is obtained by mapping a set A to the set of its limit points. The ω-coclosure of this operator is the perfect kernel of the space. The Cantor–Bendixson theorem implies in second countable spaces that the complement of the perfect kernel is the scattered part of the space and it is countable. We can use the notion of T-coclosure to study spaces of higher weight. Let us suppose E is a closed subspace of the space $\omega_1^{\omega_1}$ with the topology determined by the basic neighbourhoods

$$N(f, \alpha) = \{g \in E \mid \forall \beta < \alpha\, (f(\beta) = g(\beta))\}.$$

Let X be the tree of countable sequences of pairs (f, δ), where $f \in E$ and $\delta < \omega_1$. Let X_0 be the set of $s = \langle (f_\alpha, \delta_\alpha) \rangle_{\alpha<\beta}$ in X such that f_γ and f_ν are different but agree on δ_ν whenever $\nu < \gamma$. Let Γ_{cb} be the following monotone operator on X:

$$s \in \Gamma_{\mathrm{cb}}(A) \Leftrightarrow \exists \delta\, \forall f\, (s^\frown(f, \delta) \in X_0 \Rightarrow s^\frown(f, \delta) \in A).$$

Now it is easy to see that f is in the perfect kernel of E iff $\langle (f, 0) \rangle$ is in the ω-coclosure of Γ_{cb} and f is in the scattered part of E iff $\langle (f, 0) \rangle$ is in the ω-closure of Γ_{cb}. The ω_1-coclosure and ω_1-closure of Γ_{cb} are studied in [12]. For example, the following Cantor–Bendixson theorem is consistent relative to the consistency of a measurable cardinal: Every closed subset of $\omega_1^{\omega_1}$ is the union of its ω_1-coclosure, which is empty or of cardinality 2^{ω_1}, and its ω_1-closure, which has cardinality $\leqslant \omega_1$.

Let $G^*(\Gamma, x, T)$ be like $G(\Gamma, x, T)$ except that $\forall$ has to go up the tree T rather than $\exists$. Let

$$\mathrm{Ind}^*(\Gamma, T) = \{x \in X \mid \exists \text{ has a no-losing strategy in } G^*(\Gamma, x, T)\},$$
$$\mathrm{Coind}^*(\Gamma, T) = \{x \in X \mid \forall \text{ has a winning strategy in } G^*(\Gamma, x, T)\}.$$

Notice that if $T = \omega_1$, then there is no difference between $G^*(\Gamma, x, T)$ and $G(\Gamma, x, T)$ and so the asterisks can be dropped in this case from Ind* and Coind* in the following observation.

4.11. Lemma. (i) $\mathrm{Ind}(\Gamma^*, T) = \mathrm{Coind}^*(\Gamma, T)$.
(ii) $\mathrm{Coind}(\Gamma^*, T) = \mathrm{Ind}^*(\Gamma, T)$.

Proof. (i) To prove $\mathrm{Ind}(\Gamma^*, T) \subseteq \mathrm{Coind}^*(\Gamma, T)$, one uses the fact that if $x_\alpha \in \Gamma^*(A_\alpha)$ and $x_\alpha \in \Gamma(B_\alpha)$, then there is $x_{\alpha+1} \in A_\alpha \cap B_\alpha$. For the converse inclusion one uses the fact that if $A_{\alpha+1}$ consists of one $x_{\alpha+1} \in B$ for each B such that $x_\alpha \in \Gamma(B)$, then $x_\alpha \in \Gamma^*(A_{\alpha+1})$. The proof of (i) is similar. □

It is rather easy to see that the sets

$$\mathrm{Ind}(\Gamma, T), \quad X - \mathrm{Coind}(\Gamma, T), \quad \mathrm{Ind}^*(\Gamma, T), \quad X - \mathrm{Coind}^*(\Gamma, T)$$

are all fixed points of Γ provided that $T \leqslant \{t \in T \mid t_0 < t\}$ for all $t_0 \in T$ of height 1. A set A is *strongly Γ-dense* if there is a tree T and an onto mapping $f: T \to A$ such that

(1) Every branch of T is countable and has a last element.

(2) If $x \in T$ and B is the set of immediate successors of x in T, then $f(x) \in \Gamma(f(B))$.

Note that a strongly Γ-dense set is necessarily Γ-dense.

4.12. Lemma. $\mathrm{Ind}^*(\Gamma, \omega_1)$ *is the union of all strongly Γ-dense sets.*

Proof. Suppose $x \in \mathrm{Ind}^*(\Gamma, \omega_1)$. So $\exists$ has a no-losing strategy τ in $G^*(\Gamma, x, \omega_1)$. We get a strongly Γ-dense set containing x by considering the tree of sequences of moves in $G^*(\Gamma, x, \omega_1)$ when $\exists$ uses τ. Conversely, suppose x belongs to a strongly Γ-closed set A. Then $\exists$ has a simple no-losing strategy in $G^*(\Gamma, x, \omega_1)$ based on using the tree behind A. □

5. Some variations

The Aczel game was extended over limit steps by considering limits of ascending sequences of elements. If player $\exists$ has chosen in $G(\Gamma, x, T)$ for example X_1 so that $x \in X_1$, then $\forall$ is allowed to choose $x_1 = x$. So the sequences of elements connected to the Aczel game need not be strictly ascending. There is also an alternative definition which is based on strictly ascending sequences. Consider a modified Aczel game $G_1(\Gamma, x, T)$ where player $\exists$ has to play the sets X_α so that they contain only elements $y > x_\alpha$. In other respects the game $G_1(\Gamma, x, T)$ is defined as the Aczel game.

5.1. Lemma. *The games $G(\Gamma, x, T)$ and $G_1(\Gamma, x, T)$ are equivalent in the sense that $\exists$ has a winning strategy in one, if and only if $\exists$ has a winning strategy in the other; and that $\forall$ has a no-losing strategy in one, if and only if $\forall$ has a no-losing strategy in the other.*

Proof. In both of the games, $\exists$ can win only in a situation where $X_\alpha = \emptyset$ for some α. If $\exists$ chooses $x_\alpha \in X_\alpha$, then $\forall$ can choose $x_{\alpha+1} = x_\alpha$ which only wastes time available for $\exists$. Thus it is easy to see that $\exists$ has a winning strategy in one of the games, if and only if $\exists$ has one in the other.

If τ is a no-losing strategy of $\forall$ in $G = G(\Gamma, x, T)$, then τ (or more exactly: a restriction of it) is a no-losing strategy of $\forall$ in $G_1 = G_1(\Gamma, x, T)$. Assume then that τ_1 is a no-losing strategy of $\forall$ in G_1. We extend τ_1 to a strategy τ of $\forall$ in G by the following simulation process. Let $\exists$ play x_0 and X_0 in G. If X_0 contains only elements $y > x$, then $t_0' = t_0$ and $X_0' = X_0$ form the first move of $\exists$ in the simulated play of G_1. To this, τ_1 gives a reply $x_1' \in X_0'$, and the first move of $\forall$ in G is $x_1 = x_1'$. In case $x \in X_0$, we let $\forall$ play $x_1 = x$ in G, and more generally, $x_{\alpha+1} = x$ as long as $x \in X_\alpha$. If α is the smallest ordinal with $x \notin X_\alpha$, then we let $\exists$ play $t_0' = t_\alpha$ and $X_0' = X_\alpha$ in the simulation of G_1. Then τ gives $x_1' \in X_0'$ and we let $\forall$ play $x_{\alpha+1} = x_1'$ in G. It is easy to see that this idea can be iterated and that it produces a no-losing strategy of $\forall$ in G. Notice especially that if α is a limit ordinal and that the elements x_ν, $\nu < \alpha$ have been played in G, then the elements x_μ', $\mu < \beta$, have been played in the simulation of G_1 in such a way that $\lim_{\nu<\alpha} x_\nu = \lim_{\mu<\beta} x_\mu'$. $\square$

Next we consider a variant of the Aczel game where at limit steps, $\forall$ is allowed to play also other elements than the limit of the previously chosen ones. In this situation we assume that in $\boldsymbol{X}$ every countable ascending sequence has upper bounds, but we do not assume the existence of limits. The game $G_2(\Gamma, x, T)$ is defined in other respects like the Aczel game, but if α is a limit ordinal then x_α is allowed to be any upper bound of the x_ν where $\nu < \alpha$. If $\boldsymbol{X}$ has limits of ascending countable sequences, then it is immediate to see that if $\exists$ has a winning strategy in $G_2(\Gamma, x, T)$, then $\exists$ has one in $G(\Gamma, x, T)$, and if $\forall$ has a no-losing strategy in $G(\Gamma, x, T)$, then $\forall$ has one in $G_2(\Gamma, x, T)$.

If we consider countable sequences besides elements of X, then we can show that $G_2(\Gamma, x, T)$ gives nothing new. In later sections we shall consider definability on a structures which can code countable sequences. There the passage from X to Y makes no difference and the following result can be applied. Let $\boldsymbol{X}$ be as in the definition of $G_2(\Gamma, x, T)$. Let $\boldsymbol{Y}$ be the set of countable sequences of elements of X ordered according to the initial segment relation. We consider the following monotone operator Φ on $\boldsymbol{Y}$. If $\boldsymbol{x} = (x_\nu)_{\nu<\alpha}$, then $\boldsymbol{x} \in \Phi(B)$, if and only if either

(1) $\alpha = \beta + 1$ and $x_\beta \in \Gamma(A)$ where A is the set of all y with $\boldsymbol{x}^\frown y \in B$; or

(2) α is a limit ordinal and if $y \geq x_\nu$ for all $\nu < \alpha$, then $y \in \Gamma(A)$ where A is the set of all z with $\boldsymbol{x}^\frown \langle y, z \rangle \in B$.

The proof of the following lemma is straightforward and left to the reader.

5.2. Lemma. *The games $G_2(\Gamma, x, T)$ and $G(\Phi, \langle x \rangle, T)$ are equivalent in the sense that $\exists$ has a winning strategy in one of the games, if and only if $\exists$ has a winning strategy in the other; and that $\forall$ has a no-losing strategy in one of the games, if and only if $\forall$ has a no-losing strategy in the other.*

The referee of the first version of this paper asked how the following game is connected to the Aczel game. Let X be a set (no ordering is assumed) and Γ be monotone on X in the usual sense. The game $G_3(\Gamma, x, T)$ is like the Aczel game, but all requirements referring to the underlying ordering are deleted. Instead, it is required that $\exists$ chooses the sets $X_0, X_1, \ldots, X_\nu, \ldots$ so that $X_0 \supseteq X_1 \supseteq \cdots \supseteq X_\nu \supseteq \cdots$. Moreover, it is required that $\forall$ chooses x_α from $\bigcap_{\nu<\alpha} X_\nu$ when α is a limit ordinal. The first player who cannot move loses while the other one wins.

To cope with $G_3(\Gamma, x, T)$, we have to add more structure than in connection with $G_2(\Gamma, x, T)$. Let Γ be monotone on the set X. Let $\boldsymbol{Y}$ be the set of pairs $(\boldsymbol{x}, \boldsymbol{B})$ as below, where $\boldsymbol{x}$ is a countable sequence of elements of X and $\boldsymbol{B}$ is countable descending sequence of subsets of X; ordered by the initial segment relation on both coordinates. We require moreover that $B_0 = X$ and that $\boldsymbol{x}$ and $\boldsymbol{B}$ are of the same length. Denote $\boldsymbol{x} = (x_\nu)_{\nu<\alpha}$ and $\boldsymbol{B} = (B_\nu)_{\nu<\alpha}$. Define a monotone operator Φ on the ordered structure $\boldsymbol{Y}$ by $(\boldsymbol{x}, \boldsymbol{B}) \in \Phi(C)$, if and only if

(1) $\alpha = \beta + 1$ and there is $A \subseteq B_\beta$ with $x_\beta \in \Gamma(A)$ and for all $y \in A$, $(\boldsymbol{x}^\frown\langle y\rangle, \boldsymbol{B}^\frown\langle A\rangle) \in C$; or

(2) α is a limit ordinal and $\bigcap_{\nu<\alpha} B_\nu = \emptyset$; or

(3) α is a limit ordinal and for all $y \in \bigcap_{\nu<\alpha} B_\nu$ there is $A_y \subseteq \bigcap_{\nu<\alpha} B_\nu$ with $y \in \Gamma(A_y)$ and $(\boldsymbol{x}^\frown\langle y, z\rangle, \boldsymbol{B}^\frown\langle \bigcap_{\nu<\alpha} B_\nu, A_y\rangle) \in C$ for all $z \in A_y$.

The following lemma is easy to prove.

5.3. Lemma. *The games $G_3(\Gamma, x, T)$ and $G(\Phi, (\langle x\rangle, \langle X\rangle), T)$ are equivalent in the sense that $\exists$ has a winning strategy in one, if and only if $\exists$ has a winning strategy in the other; and $\forall$ has a no-losing strategy in one, if and only if $\forall$ has a no-losing strategy in the other.*

We have here sketched how some variants of the Aczel game can be reduced to the Aczel game. It is left as an exercise to the reader to play with reductions the other way round.

6. T-inductive definability

Let $\mathfrak{A}$ be a first-order structure for a language L. We assume that $\boldsymbol{X}$ is part of the structure of $\mathfrak{A}$, that is,

$$\mathfrak{A} = \langle A, \ldots, X, \leqslant, \ldots\rangle.$$

We say that $\mathfrak{A}$ is a structure *around* $\boldsymbol{X}$. We include the case that X is an n-ary relation on A. In that case we say that $\boldsymbol{X}$ is *n-ary*. Recall that $\boldsymbol{X} = \langle X, \leqslant\rangle$ is supposed to be a partially ordered structure in every countable ascending sequence $(x_\alpha)_{\alpha<\nu}$ has a unique supremum $\lim_{\nu<\alpha} x_\alpha$.

We make throughout this section the assumption that the tree T is *reflexive*, i.e., that $T \leqslant T_t$ holds for all $t \in T$, where T_t denotes the subtree $\{t' \in T : t \leqslant t'\}$.

This assumption is needed in 6.3 and 6.4. Notice that the tree ω_1 consisting of one single ω_1-branch is reflexive. Every tree T can be extended to a reflexive tree by iterating it in the following way (see [3] and [5]). Let $R(T)$ be the set of finite sequences $(t_0, \ldots, t_n)$ of elements of T. We can think of this sequence as a linear ordering which starts with $\{t \in T: t \leqslant t_0\}$, continues with $\{t \in T: t \leqslant t_1\}$, then with $\{t \in T: t \leqslant t_2\}$, etc. until t_n comes in the end. In this way $R(T)$ gets a natural tree-ordering: if s and s' are elements of $R(T)$, then we define $s \leqslant s'$ to mean that as linear orderings, s is equal to s' or is an initial segment of s'. It is easy to see that $T \leqslant R(T)$ and that $R(T)$ is reflexive. It is also interesting to note that if T has no branches of length $\kappa > \omega$, then neither has $R(T)$. We can split $R(T)$ into parts that are called *phases*. Namely, if $s = (t_0, \ldots, t_n) \in R(T)$, we call the number n the phase of s and denote it by $p(s)$. Elements of phase 0 form an isomorphic copy of T. Each element $(t_0, \ldots, t_n)$ of phase n extends to an isomorphic copy $\{(t_0, \ldots, t_{n+1}): t_{n+1} \in T\}$ of T.

Any first-order formula $\phi(x_1, \ldots, x_n, S)$ of the language $L \cup \{S\}$ with S positive determines a monotone operator Γ_ϕ on X as follows:

$$(x_1, \ldots, x_n) \in \Gamma_\phi(B) \Leftrightarrow \mathfrak{A} \models \phi(x_1, \ldots, x_n, B).$$

An operator Γ on X is called *positive elementary on* $\mathfrak{A}$ if there is an S-positive formula ϕ with $\Gamma = \Gamma_\phi$.

Definition. An n-ary relation R on $\mathfrak{A}$ is called *T-inductive* if there is a positive elementary operator Γ on $\mathfrak{A}$ and elements $a_1, \ldots, a_m$ of A such that

$$R(x_1, \ldots, x_n) \Leftrightarrow (x_1, \ldots, x_n, a_1, \ldots, a_m) \in \operatorname{Ind}(\Gamma, T).$$

An n-ary relation R on $\mathfrak{A}$ is called *T-coinductive* if there is a positive elementary operator Γ on $\mathfrak{A}$ and elements $a_1, \ldots, a_m$ of A such that

$$R(x_1, \ldots, x_n) \Leftrightarrow (x_1, \ldots, x_n, a_1, \ldots, a_m) \in \operatorname{Coind}(\Gamma, T).$$

The notion *T-inductive* (or *T-coinductive*) in $Q_1, \ldots, Q_n$ is defined as above using $Q_1, \ldots, Q_n$ as positive (or respectively negative) parameters. A second-order relation $\boldsymbol{R}$ of sequences $\langle x_1, \ldots, x_n, R_1, \ldots, R_m\rangle$, where x_i are elements of A and R_i are relations on A, is *T-inductive* (or *T-coinductive*), if the relation $\{\langle x_1, \ldots, x_n\rangle \mid \langle x_1, \ldots, x_n, R_1, \ldots, R_m\rangle \in \boldsymbol{R}\}$ is *T-inductive* or *T-coinductive* as above with the relations $R_1, \ldots, R_m$ as positive (or, respectively, negative) parameters. If T is an ordinal $\langle \alpha, <\rangle$, we use 'α-inductive' for 'T-inductive' and 'α-coinductive' for 'T-coinductive'.

6.1. Theorem. *Let* $\mathfrak{A}$ *be a structure around* $\boldsymbol{X}$. *Let* $\boldsymbol{WF}_1$ *be the set of subsets of* X *which contain no uncountable ascending chains. Then* $\boldsymbol{WF}_1$ *is* ω_1*-inductive on* $\mathfrak{A}$.

Proof. We imitate the proof of Theorem 6A.1. of [8]. Let c and d be two distinct elements of X. Let $\phi(t, x, T, S)$ be the formula:

$$\{t = c \wedge \forall y \in T\,[(x < y \wedge \neg\exists z\,(x < z < y)) \rightarrow (c, y) \in S]\}$$
$$\vee \{t = d \wedge \forall y \in T\,[(c, y) \in S]\}.$$

Let us fix T and let $R(t, x) \Leftrightarrow (t, x, T) \in \Gamma^{\omega_1}$. We claim that T has no uncountable branches iff $R(d, d)$. So suppose first T has no uncountable branches. We describe the winning strategy of $\exists$ in $G(\Gamma_\phi, (d, d), \omega_1)$. In this case there is no need to specify the moves in ω_1, as long as they go up. Player $\exists$ starts with $A_0 = \{c\} \times T$. If $\forall$ has played (c, x_α), $\exists$ lets A_α to be the set of pairs (c, y) where y is a successor of x_α in T. Since T has no uncountable branches, a moment comes when $A_\alpha = \emptyset$ and $\exists$ has won. For the converse, suppose T has an uncountable branch b. Now $\forall$ has an easy no-losing strategy which is based on following b. $\square$

Theorem 6.1 is in fact a special case of the following more general result, which we quote without proof.

6.2. Theorem. *Let $\mathfrak{A}$ be a structure around $\boldsymbol{X}$. Let T be a tree. Then the set of suborderings U of $\boldsymbol{X}$ which are trees and which satisfy $U \ll T$, is T-inductive on $\mathfrak{A}$. The set of suborderings U of $\boldsymbol{X}$ which are trees and which satisfy $T \leqslant U$, is T-coinductive on $\mathfrak{A}$.*

The following Transitivity Lemma is adapted from [8]. We have a first-order formula ϕ and the associated operator Γ_ϕ which is used to define a T-inductive relation R. The point is that ϕ has a predicate S which occurs positively and is itself T-inductive on $\mathfrak{A}$. The lemma shows that S can be eliminated from the definition of R.

6.3. Transitivity Lemma. *Suppose R is T-inductive in S and $Q_1, \ldots, Q_n$, and S is T-inductive in $Q_1, \ldots, Q_n$. Suppose moreover that T is reflexive. Then R is T-inductive in $Q_1, \ldots, Q_n$. The same is true of T-coinductive relations.*

Proof. The Transitivity Lemma follows easily from the following Combination Lemma as in the classical case considered in [8]. Let $\psi(\boldsymbol{u}, \boldsymbol{y}, S, \boldsymbol{V})$ be a formula in which S and $\boldsymbol{V}$ occur only positively, and $\phi(\boldsymbol{x}, S, Q, \boldsymbol{V})$ a formula in which S, Q, $\boldsymbol{V}$ occur only positively. The predicates of $\boldsymbol{V}$ are parameters throughout. In (i) and (ii) below,

$$S(\boldsymbol{y}) \Leftrightarrow (\boldsymbol{a}, \boldsymbol{y}) \in \mathrm{Ind}(\Gamma_\psi, T),$$

and in (iii) and (iv),

$$S(\boldsymbol{y}) \Leftrightarrow (\boldsymbol{a}, \boldsymbol{y}) \notin \mathrm{Coind}(\Gamma_\psi, T).$$

Let $\boldsymbol{a}$, $\boldsymbol{u}^*$, $\boldsymbol{y}^*$ and $\boldsymbol{x}^*$ be fixed parameter sequences and c and d two distinct elements.

Claim (Combination Lemma). *There is a formula $\vartheta(t, \boldsymbol{u}, \boldsymbol{y}, \boldsymbol{x}, U, V)$ for which*

(i) $(\boldsymbol{u}, \boldsymbol{y}) \in \mathrm{Ind}(\Gamma_\psi, T) \Leftrightarrow (c, \boldsymbol{u}, \boldsymbol{y}, \boldsymbol{x}^*) \in \mathrm{Ind}(\Gamma_\vartheta, T)$,

(ii) $\boldsymbol{x} \in \mathrm{Ind}(\Gamma_\phi, T) \Leftrightarrow (d, \boldsymbol{u}^*, \boldsymbol{y}^*, \boldsymbol{x}) \in \mathrm{Ind}(\Gamma_\vartheta, T)$,

(iii) $(\boldsymbol{u}, \boldsymbol{y}) \in \mathrm{Coind}(\Gamma_\psi, T) \Leftrightarrow (c, \boldsymbol{u}, \boldsymbol{y}, \boldsymbol{x}^*) \in \mathrm{Coind}(\Gamma_\vartheta, T)$,

(iv) $\boldsymbol{x} \in \mathrm{Coind}(\Gamma_\phi, T) \Leftrightarrow (d, \boldsymbol{u}^*, \boldsymbol{y}^*, \boldsymbol{x}) \in \mathrm{Coind}(\Gamma_\vartheta, T)$.

Proof. Let $\vartheta(t, \boldsymbol{u}, \boldsymbol{y}, \boldsymbol{x}, U, V)$ be the following formula:

$$[t = c \wedge \psi(\boldsymbol{u}, \boldsymbol{y}, \{(\boldsymbol{u}', \boldsymbol{y}'): U(c, \boldsymbol{u}', \boldsymbol{y}', \boldsymbol{x}^*)\}, V)]$$
$$\vee [t = d \wedge \phi(\boldsymbol{x}, \{\boldsymbol{y}': U(c, \boldsymbol{a}, \boldsymbol{y}', \boldsymbol{x}^*)\}, \{\boldsymbol{x}': U(d, \boldsymbol{u}^*, \boldsymbol{y}^*, \boldsymbol{x}')\}, V)].$$

The assertion follows by comparing the games corresponding to the operations appearing in the equivalences. We consider here the coclosures only. Assertion (iii) is trivial because both sides of the equivalence are defined by essentially the same game. For the second equivalence, assume $\boldsymbol{x} \notin \mathrm{Coind}(\Gamma_\phi, T)$. Then $\forall$ has no no-losing strategy in the game $G(\Gamma_\phi, x, T)$. Also, $\forall$ has no no-losing strategy in the game $G(\Gamma_\psi, (\boldsymbol{a}, \boldsymbol{y}), T)$ for any $\boldsymbol{y} \in S$. Let us then consider the game $G(\Gamma_\vartheta, (d, \boldsymbol{u}^*, \boldsymbol{y}^*, \boldsymbol{x}), T)$. Define for any U,

$$U^0 = \{\boldsymbol{y}: U(c, \boldsymbol{a}, \boldsymbol{y}, \boldsymbol{x}^*)\},$$
$$U^1 = \{\boldsymbol{x}': U(d, \boldsymbol{u}^*, \boldsymbol{y}^*, \boldsymbol{x}')\}.$$

In order to demonstrate that $\forall$ cannot have a no-losing strategy, we let $\exists$ play with $U^0 = S$ as long as $\forall$ plays with sequences where $t = d$. A strategy of $\forall$ where $t = d$ is always chosen is essentially a strategy of $\forall$ in the game $G(\Gamma_\phi, \boldsymbol{x}, T)$ by our convention on the behaviour of $\exists$. Therefore it cannot be a no-losing strategy. On the other hand, if $\forall$ chooses $t = c$, then the subsequent part of $G(\Gamma_\vartheta, (d, \boldsymbol{u}^*, \boldsymbol{y}^*, \boldsymbol{x}), T)$ is essentially one of the games $G(\Gamma_\psi, (\boldsymbol{a}, \boldsymbol{y}), T)$, $\boldsymbol{y} \in S$, whence $\forall$ cannot have a no-losing strategy there either. The converse implication is proved in a similar way. □

6.4. Corollary. *Suppose that T is reflexive. The class of relations T-inductive in $Q_1, \ldots, Q_n$ is closed under $\cup$, $\cap$, $\forall$ and $\exists$.*

7. Relation to $\boldsymbol{\Sigma}_1^1$-definability

A relation $R(x_1, \ldots, x_n)$ on A is *$\boldsymbol{\Sigma}_1^1$-definable on* $\mathfrak{A}$ if there is an elementary formula $\phi(x_1, \ldots, x_n, y_1, \ldots, y_m, Q_1, \ldots, Q_m)$ and $a_1, \ldots, a_m$ in A such that

$$R(x_1, \ldots, x_n)$$
$$\Leftrightarrow \mathfrak{A} \models \exists Q_1, \ldots, Q_m\, \phi(x_1, \ldots, x_n, a_1, \ldots, a_m, Q_1, \ldots, Q_m).$$

We say that $\mathfrak{A}$ *codes countable sequences* if $\mathfrak{A}$ has definable subsets Ω and Seq, a definable relation $<$ and definable functions $q(x, y)$ and $\mathrm{lh}(x)$ such that for some $\pi: \langle \omega_1, < \rangle \cong \langle \Omega, < \rangle$ it holds that for every countable sequence $(a_\beta)_{\beta<\alpha}$ from A there is some $a \in \Omega$ with $a_\beta = q(a, \pi(\beta))$ for all $\beta < \alpha$ and $\mathrm{lh}(a) = \pi(\alpha)$. In such a case there is a natural definable tree-ordering on $\mathfrak{A}$: the tree of codes of countable sequences of elements of $\mathfrak{A}$. We denote this tree by $\boldsymbol{X}_{\mathfrak{A}}$. The structure $(HC, \in)$ is an example of a structure that codes countable sequences.

An operator $\Gamma: \mathcal{P}(X) \to \mathcal{P}(X)$ is *nice* if $x \in \Gamma(A)$ implies $x \in \Gamma(A')$ for some set A' of immediate successors of x. A relation is *nicely* T-inductive (T-coinductive) if it satisfies the definition of T-inductive (respectively, T-coinductive) with a nice positive elementary operator. Typically, cases where the elements of $\boldsymbol{X}$ can be understood as sequences yield examples of nice operators. A very important special case is the operator connected to the Ehrenfeucht–Fraïssé game which was discussed in Example 2.3. On the other hand, one can consider among all linear orderings the class of well-orderings or that of κ-*well-orderings*, i.e., the class of all linear orderings which contain no descending sequences of cardinality κ. There is a monotone operator Γ which, intuitively, accepts a linear ordering whenever all of its proper initial segments have been accepted. The ω-closure of this operator is the class of all linear orderings and the ω_1-closure is that of all ω_1-well-orderings. It is easy to see that this operator Γ is not nice. (For more about this operator, see [10].)

7.1. Theorem. *Suppose $\mathfrak{A}$ is a structure around a tree $\boldsymbol{X}$ and $\mathfrak{A}$ codes countable sequences. Then every nicely ω_1-coinductive relation on $\mathfrak{A}$ is $\boldsymbol{\Sigma}^1_1$-definable on $\mathfrak{A}$.*

Proof. Suppose $\phi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ is an S-positive formula and $a_1, \ldots, a_m$ are in A such that

$$R(x_1, \ldots, x_n) \Leftrightarrow (x_1, \ldots, x_n, a_1, \ldots, a_m) \in \mathrm{Coind}(\Gamma_\phi, \omega_1).$$

Let us assume for simplicity that $n = 1$ and there are no parameters $a_1, \ldots, a_m$. It suffices to prove that $R(x)$ is equivalent to the following condition

(*) There is a subset P of A such that $\Gamma(P) \subseteq P$, $x \notin P$ and $-P$ is closed under limits of ascending ω-sequences.

Suppose first $R(x)$. Let τ be a no-losing strategy of $\forall$ in $G(\Gamma, x, \omega_1)$. Let Φ be the dual of Γ, that is, $\Phi(A) = -\Gamma(-A)$. Player $\exists$ has the following no-losing strategy in $G(\Phi, x, \omega_1)$. While he plays $G(\Phi, x, \omega_1)$, he simulates a play of $G(\Gamma, x, \omega_1)$ as well, letting $\forall$ follow τ. Whenever $\forall$ has played x_α in $G(\Gamma, x, \omega_1)$, $\exists$ takes one τ-move $x_{\alpha+1} \in A$ for each A such that $x_\alpha \in \Gamma(A)$ and A is a set of immediate successors of x_α in $\boldsymbol{X}$, and lets A_α consists of the chosen elements $x_{\alpha+1}$. This is the move of $\exists$ in $G(\Phi, x, \omega_1)$. The next move of $\forall$ in $G(\Phi, x, \omega_1)$ determines a set A which is the next move of $\exists$ in the auxiliary game $G(\Gamma, x, \omega_1)$. Let σ denote this no-losing strategy. Note that this strategy forces $\forall$ to play elements which are immediate successors of each other. Let Q be the set of moves of $\forall$ in various rounds of the game $G(\Phi, x, \omega_1)$ when $\exists$ plays σ. Let H be the union of Q and the set of limits of ascending ω-sequences of elements of Q. Let (x_n) be one such ω-sequence. So x_n is (either x or) a move number α_n of $\forall$ in some round of the game $G(\Phi, x, \omega_1)$. There is some sequence x^n_β, $\beta \leq \alpha_n$, of moves of $\forall$ in $G(\Phi, x, \omega_1)$ which leads to x_n. Since $\boldsymbol{X}$ is a tree and successive moves of $\forall$ in $G(\Phi, x, \omega_1)$ are successors of each other in $\boldsymbol{X}$, x_m $(m<n)$ is an element of the sequence x^n_β, $\beta<\alpha_n$, and indeed $x^m_\beta = x^n_\beta$ for $m<n$ and $\beta<\alpha_m$. This means that the sequence (x_n) is part of a sequence of moves of $\forall$ during one single round of $G(\Phi, x, \omega_1)$. This shows that elements of H are moves of $\forall$ or moves that arise at limit stages of rounds of the game $G(\Phi, x, \omega_1)$. Let P be the complement of $\{x\} \cup H$. Certainly $x \notin P$. To prove $\Gamma(P) \subseteq P$, suppose $t \in \Gamma(P) - P$. Now σ gives a set A with $t \in \Phi(A)$. Since $t \in \Gamma(P)$, but $t \notin \Gamma(-A)$, there is some t' in $P \cap A$. But $P \cap A = \emptyset$ and we have a contradiction. Finally, $-P$ is by construction closed under limits of countable ascending sequences.

For the converse, suppose a set P satisfying $(*)$ is found. Now $\forall$ has a simple no-losing strategy in $G(\Gamma, x, \omega_1)$. He just has to keep his moves out of P. Suppose $\forall$ has played x_α and $\exists$ answers with A_α such that $x_\alpha \in \Gamma(A_\alpha)$. Since $x_\alpha \notin P$, there is some $x_{\alpha+1} \in A_\alpha - P$. The limit stages present no problem since $-P$ is closed under limits. $\square$

7.2. Theorem. *Suppose $\mathfrak{A}$ is a structure of cardinality ω_1 which codes countable sequences. Then every $\boldsymbol{\Sigma}^1_1$-definable relation on $\mathfrak{A}$ is nicely ω_1-coinductive on $\mathfrak{A}$ relative to the partial ordering $\boldsymbol{X}_{\mathfrak{A}}$.*

Proof. The proof is built on the proof of Theorem 8A.1 in [8]. For simplicity, let us assume we have a unary relation $R(x)$ on $\mathfrak{A}$ whith the following definition

$$R(x) \Leftrightarrow \exists Q_1 \cdots \exists Q_n\, \phi(x, Q_1, \ldots, Q_n)$$

where ϕ is first-order. We assume here that the language is (made) relational. As observed in [8], we can eliminate quantifiers from ϕ by adding new relations Q_i. So we can assume without loss of generality that $\phi(x, Q_1, \ldots, Q_n)$ is of the form

$$\forall z_1 \cdots z_k\, \exists y_1 \cdots \exists y_j\, \psi(x, z_1, \ldots, z_k, y_1, \ldots, y_j, Q_1, \ldots, Q_n)$$

where ψ is quantifier-free. To further simplify notation we assume $k=j=1$. So finally

$$R(x) \Leftrightarrow \exists Q_1 \cdots \exists Q_n\, \forall z\, \exists y\, \psi(x, z, y, Q_1, \ldots, Q_n).$$

Let L be the language $\{Q_1, \ldots, Q_n\}$. The L-structures below are all assumed to have a subset of A as their universe. Fix $x \in A$. If $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are L-structures, we write $\mathfrak{B}_1 < \mathfrak{B}_2$ provided that $\mathfrak{B}_1$ is a substructure of $\mathfrak{B}_2$, $x \in B_1$ and for all z in B_1, $\mathfrak{B}_2 \models \exists y\, \psi(x, z, y, Q_1, \ldots, Q_n)$.

Let $G_1(x)$ be the following game. There are two players $\forall$ and $\exists$ and ω_1 moves.

$\forall$	$\exists$	conditions
	$\mathcal{B}_0$	$x \in B_0$
d_0		
	$\mathcal{B}_1$	$\mathcal{B}_0 < \mathcal{B}_1$ and $d_0 \in B_1$
d_1		
	$\mathcal{B}_2$	$\mathcal{B}_1 < \mathcal{B}_2$ and $d_1 \in B_2$
.	.	.
.	.	.
.	.	.
	$\mathcal{B}_\nu$	$\nu = \cup\nu$ and $\bigcup_{\alpha<\nu}\mathcal{B}_\alpha < \mathcal{B}_\nu$
d_ν	.	.
.	.	.
.	.	.

Fig. 2. The game $G_1(x)$.

Player $\forall$ plays elements of A and $\exists$ plays countable L-structures $\mathfrak{B}_\alpha$. The rules of the game are given in Fig. 2. Player $\exists$ wins $G_1(x)$ if he can make all his ω_1 moves. It is relatively easy to see (and essentially proved in [8, p. 134]) that $R(x)$ holds iff $\exists$ has a winnning strategy in $G_1(x)$.

Using the assumed coding of countable sequences it is possible to define the following predicate $\mathrm{Str}(x, w)$ on $\mathfrak{A}$:

"w is a sequence $(w_\beta)_{\beta<\alpha}$, $\alpha < \omega_1$, where w_β is a pair $(\mathfrak{B}_\beta, d_\beta)$ such that $\mathfrak{B}_\beta$ is an L-structure and $d_\beta \in B_\beta$ and $\mathfrak{B}_\gamma < \mathfrak{B}_\beta$ for $\gamma < \beta < \alpha$".

Let $\theta(x, w, S)$ be a first-order formula expressing on $\mathfrak{A}$ the following:

"w is a sequence $(w_\beta)_{\beta<\alpha}$, $\alpha < \omega_1$, where w_β is a pair $(\mathfrak{B}_\beta, d_\beta)$ such that $\mathfrak{B}_\beta$ is an L-structure, and if $\mathrm{Str}(x, w)$ holds, then for every countable L-structure $\mathfrak{B}$ there is an element d such that $w' \in S$ where w' is the extension of w by $(\mathfrak{B}, d)$".

Let $G_2(x)$ be the game $G(\emptyset, \Gamma_\theta, \omega_1)$.

Claim. *$R(x)$ holds iff $\forall$ has a no-losing strategy in $G_2(x)$.*

Proof. Suppose first that $R(x)$ holds and that τ is a winning strategy of $\exists$ in $G_1(x)$. We describe a no-losing strategy of $\forall$ in $G_2(x)$. Let $\mathfrak{B}_0$ be the opening τ-move of $\exists$ in $G_1(x)$. In the beginning of $G_2(x)$ $\exists$ plays A_0 with $w_0 = \emptyset \in \Gamma_\theta(A_0)$. Now $\mathrm{Str}(x, w_0)$ is true, whence there is an element d_1 such that $\langle(\mathfrak{B}_0, d_1)\rangle \in A_0$. The pair $x_1 = (\mathfrak{B}_0, d_1)$ is the first move of $\forall$ in $G_2(x)$. Suppose then $\exists$ has played $\mathfrak{B}_{\alpha+1}$ in $G_1(x)$ and $A_{\alpha+1}$ in $G_2(x)$ with $x_{\alpha+1} \in \Gamma_\theta(A_{\alpha+1})$. By the definition of θ there is a $d_{\alpha+2}$ such that $x_{\alpha+2} \in A_{\alpha+1}$, where $x_{\alpha+2}$ codes the extension of the sequence coded by $x_{\alpha+1}$ by $(\mathfrak{B}_{\alpha+1}, d_{\alpha+2})$. Let us finally consider a limit stage ν of the game. The element x_ν is necessarily the supremum of $(x_\alpha)_{\alpha<\nu}$. Suppose $\exists$ produces A_ν with $x_\nu \in \Gamma(A_\nu)$. The strategy τ gives $\exists$ some $\mathfrak{B}_\nu$ in $G_1(x)$. By the definition of θ, there is some d_ν with $x_{\nu+1} \in A_\nu$, where $x_{\nu+1}$ is the extension of x_ν with the pair $(\mathfrak{B}_\nu, d_\nu)$. This ends the description of the no-losing strategy of $\forall$ in $G_2(x)$.

For the converse, suppose $\forall$ has a no-losing strategy τ in $G_2(x)$. We shall describe a winning strategy of $\exists$ in $G_1(x)$. The first move of $\exists$ in $G_1(x)$ is defined as follows. Let first Y_0 be the set of one-element sequences $w = \langle(\mathfrak{B}, d)\rangle$ which either satisfy $\neg\mathrm{Str}(x, w)$ or fail to be τ-moves of $\forall$ in $G_2(x)$ after some first move A_0 of $\exists$.

Case 1: $\emptyset \in \Gamma(Y_0)$. Now we can let $\exists$ start $G_2(x)$ with Y_0 and τ gives some response $w = \langle(\mathfrak{B}, d)\rangle \in Y_0$. By construction $\neg\mathrm{Str}(x, w)$. But now $\exists$ beats $\forall$ on the next move since $w \in \Gamma(\emptyset)$. So this case is impossible.

Case 2: $\emptyset \notin \Gamma(Y_0)$. Since $\mathrm{Str}(\emptyset)$, there is a $\mathfrak{B}_0$ such that for all d_0, $w_0 = \langle(\mathfrak{B}_0, d_0)\rangle \notin Y_0$. So we have both $\mathrm{Str}(x, w_0)$ and w_0 is a τ-response of $\forall$ to some move A_0 of $\exists$.

The structure $\mathfrak{B}_0$ given by Case 2 above is now the first move of $\exists$ in $G_1(x)$. Suppose $\forall$ answers with d_0. We start $G_2(x)$ by letting $\exists$ play the set A_0 given by Case 2 above and $\forall$ the element $w_0 = \langle(\mathfrak{B}_0, d_0)\rangle$. The game goes on like this and $\exists$ wins.

The Claim is now proved and thereby the whole theorem. □

7.3. Corollary. *If $\mathfrak{A}$ is a structure of cardinality ω_1 which codes countable sequences, then $\boldsymbol{\Sigma}^1_1$-definable relations on $\mathfrak{A}$ are exactly the nicely ω_1-coinductive relations on $\mathfrak{A}$ relative to the partial ordering $\boldsymbol{X}_{\mathfrak{A}}$.*

7.4. Example. Assume CH and take as $\mathfrak{A}$ the structure $\mathfrak{R} = \langle \omega \cup \omega^\omega, +, \cdot, \omega, <, 0, 1, \mathrm{Ap}\rangle$, where $\mathrm{Ap}(f, n, m)$ holds if and only if $f \in \omega^\omega$, $n \in \omega$ and $f(n) = m$. Then the $\boldsymbol{\Sigma}^1_1$-relations are exactly the nicely ω_1-coinductive relations on $\mathfrak{R}$.

8. Stages of induction

Let Γ be a monotone operator on $\boldsymbol{X}$. The smallest fixed point $\mathrm{Ind}(\Gamma, \omega)$ has a representation as a union of *stages*:

$$\mathrm{Ind}(\Gamma, \omega) = \bigcup_\alpha \mathrm{Ind}(\Gamma, B_\alpha).$$

Moreover, if $x \in \mathrm{Ind}(\Gamma, \omega)$, there is a unique α such that $x \in \mathrm{Ind}(\Gamma, \sigma B_\alpha)$ but $x \notin \mathrm{Ind}(\Gamma, B_\alpha)$. We shall prove a similar result for arbitrary T-closures $\mathrm{Ind}(\Gamma, T)$ and T-coclosures $\mathrm{Coind}(\Gamma, T)$. However, we do not get similar uniqueness as that of the α above.

Suppose Γ is a monotone operator on $\boldsymbol{X}$, x an element of X and T is a tree. We use $|x|_\Gamma^T$ to denote the tree of all pairs (τ, b) where b is a chain in T with a last element and τ is a no-losing strategy of $\forall$ in $G(\Gamma, x, b)$. These pairs are ordered as follows: $(\tau, b) \leqslant (\tau', b')$ iff b is an initial segment of b' and τ agrees with τ' as long as $\exists$ stays in b. Notice that

$$|x|_\Gamma^T \leqslant T \quad \text{and} \quad T_0 \leqslant T_1 \Rightarrow |x|_\Gamma^{T_0} \leqslant |x|_\Gamma^{T_1}.$$

For the trees B_α we have that $|x|_\Gamma^\omega \equiv B_\alpha$ whenever $x \in \mathrm{Ind}(\Gamma, B_{\alpha+1}) - \mathrm{Ind}(\Gamma, B_\alpha)$.

In the following result it should be kept in mind that $\mathrm{Ind}'(\Gamma, T) = X - \mathrm{Coind}(\Gamma, T)$ is equal to $\mathrm{Ind}(\Gamma, T)$ if Γ is open determined on T.

8.1. Proposition. *Let Γ be a monotone operator on $\boldsymbol{X}$, x an element of X and T a tree. Then*

$$x \in \mathrm{Ind}'(\Gamma, T) \Leftrightarrow |x|_\Gamma^T < T.$$

Proof. Suppose $x \in \mathrm{Coind}(\Gamma, T)$. We prove $T \leqslant |x|_\Gamma^T$. Let $t \in T$ and $b = \{t' \in T \mid t' \leqslant t\}$. The no-losing strategy of $\forall$ in $G(\Gamma, x, T)$ gives rise to a no-losing strategy τ of $\forall$ on $G(\Gamma, x, b)$. Now the mapping $g(t) = (\tau, b)$ demonstrates $T \leqslant |x|_\Gamma^T$. For the converse, suppose there is an order-preserving mapping $f: T \to |x|_\Gamma^T$. Player $\forall$ has the following no-losing strategy in $G(\Gamma, x, T)$: Whenever $\exists$ moves in T, $\forall$ uses f to find an extension of his current strategy. $\square$

8.2. Remark. The proof of Proposition 8.1 actually gives $x \in \mathrm{Coind}(\Gamma, T)$ whenever $T \leqslant |x|_\Gamma^U$ for some U.

8.3. Corollary. *Let Γ be a monotone operator on $\boldsymbol{X}$, x an element of X and T a tree with no uncountable branches. If $U = |x|_\Gamma^T$, then $x \in \mathrm{Coind}(\Gamma, T)$, but $x \notin \mathrm{Coind}(\Gamma, \sigma U)$.*

Proof. In view of Proposition 8.1 it suffices to recall that $U < \sigma U$. $\square$

8.4. Corollary. *Let Γ be a monotone operator on $\boldsymbol{X}$ and T a tree such that $U < T$ implies $\sigma U < T$ for all U. Then*

$$\mathrm{Coind}(\Gamma, T) = \bigcap \{\mathrm{Coind}(\Gamma, U) \mid U < T\}.$$

Notice that if $T = \omega$, then the trees U with $U < T$ correspond to ordinals. Also, if $T = \omega_1$, then $U < T$ iff $\sigma U < T$ iff U has no uncountable branches. Thus

$$\mathrm{Coind}(\Gamma, \omega_1) = \bigcap \{\mathrm{Coind}(\Gamma, U) \mid U \text{ has no uncountable branches}\}.$$

If $\boldsymbol{X}$ and Γ_s are as in Example 4.2 and $T \in X$, then Lemma 4.5 and the above Proposition 8.1 imply $T \leqslant |T|_{\Gamma_s}^T \leqslant T$.

8.5. Corollary. *Let Γ be monotone on* $\boldsymbol{X}$ *and Φ monotone on* $\boldsymbol{Y}$, *and let $x \in X$ and $y \in Y$. If $x \in \mathrm{Ind}'(\Gamma, T)$. Then the following conditions are equivalent*:

(1) $|x|_\Gamma^T \leq |y|_\Phi^T$.

(2) *For all $V < T$, if $x \in \mathrm{Coind}(\Gamma, V)$ then $y \in \mathrm{Coind}(\Phi, V)$.*

(3) $y \in \mathrm{Coind}(\Phi, |x|_\Gamma^T)$.

Proof. (1)$\Rightarrow$(2). Suppose $V < T$ and $x \in \Gamma_V$. By 7.1, $V \leq |x|_\Gamma^V$. Hence by (1), $V \leq |y|_\Phi^T$, whence again by 8.1, $y \in \mathrm{Coind}(\Phi, V)$.

(2)$\Rightarrow$(3). Since $x \in \mathrm{Ind}'(\Gamma, T)$, we have $|x|_\Gamma^T < T$. By 8.3 and (2), $y \in \mathrm{Coind}(\Phi, |x|_\Gamma^T)$.

(3)$\Rightarrow$(1). By 8.1, $|x|_\Gamma^T \leq |y|_\Phi^{|x|_\Gamma^T} \leq |y|_\Phi^T$. $\square$

8.6. Proposition. *Let Γ be a monotone operator on* $\boldsymbol{X}$, *x an element of X and T a tree. Then the following conditions are equivalent*:

(1) $x \in \mathrm{Ind}(\Gamma, T)$.

(2) *There is a tree $U \ll T$ such that $x \in \mathrm{Ind}(\Gamma, \sigma U)$ but $x \notin \mathrm{Ind}(\Gamma, U)$.*

Proof. Suppose $x \in \mathrm{Ind}(\Gamma, T)$. Let τ be a winning strategy of $\exists$ in $G(\Gamma, x, T)$. Let V be the tree of sequences of successor length of moves of $\forall$ in rounds of $G(\Gamma, x, T)$ when $\exists$ plays τ and $\forall$ has not lost yet. Now $\sigma V \leq T$ and $\exists$ has a winning strategy in $G(\Gamma, x, \sigma V)$. Let U be a $\ll$-minimal tree such that either $U = V$ or $U \ll V$ and $\exists$ has a winning strategy in $G(\Gamma, x, \sigma U)$. So $x \in \mathrm{Ind}(\Gamma, \sigma U)$. To prove $x \notin \mathrm{Ind}(\Gamma, U)$, assume the contrary. By starting again with U in place of T, we end up with W such that $W \ll U$ and $\exists$ has a winning strategy in $G(\Gamma, x, \sigma W)$. This is contrary to the minimality of U. $\square$

8.7. Corollary. *Let Γ be a monotone operator on* $\boldsymbol{X}$ *and T a tree. Then*

$$\mathrm{Ind}(\Gamma, T) = \bigcup \{\mathrm{Ind}(\Gamma, \sigma U) \mid U \ll T\}.$$

Proof. If $U \ll T$, i.e., $\sigma U \leq T$, then $\mathrm{Ind}(\Gamma, U) \subseteq \mathrm{Ind}(\Gamma, T)$. Conversely, if $x \in \Gamma^U$, then Proposition 8.4 gives a tree $U \ll T$ with $x \in \mathrm{Ind}(\Gamma, \sigma U)$. $\square$

Again, if $T = \omega$, then the trees U with $U \ll T$ are essentially just the trees B_α. Also, if $T = \omega_1$, then $U \ll T$ iff $\sigma U \ll T$ iff U has no uncountable branches. Thus

$$\mathrm{Ind}(\Gamma, \omega_1) = \bigcup \{\mathrm{Ind}(\Gamma, U) \mid U \text{ has no uncountable branches}\}.$$

If $T = \omega$, then for any x the trees U of Proposition 8.6 coincide (up to order-preserving mappings) with the tree $|x|_\Gamma^T$. This is not true in general as the following example shows.

8.8. Example. Let $\boldsymbol{X}$ and Γ_A be as in Example 4.8. Let $A \subset \omega_1$ be bistationary. Then $\emptyset \in \mathrm{Ind}(\Gamma_A, \sigma T(A)) - \mathrm{Ind}(\Gamma_A, T(A))$, but $T(A) \not\equiv |\emptyset|_{\Gamma_A}^{(\omega_1, <)}$. The first claim follows from the remarks made in Example 4.8. The second claim follows from the observation (proved in [6]) that $\forall$ cannot have a no-losing strategy of length $\geqslant \omega + 1$ in $G(\Gamma_A, \emptyset, \omega_1)$.

By definition, $\mathrm{Ind}(\Gamma, T) \cap \mathrm{Coind}(\Gamma, T) = \emptyset$. Hence, if $x \in \mathrm{Ind}(\Gamma, T)$, Proposition 8.1 gives $|x|_\Gamma^T < T$. We can actually get a bit more:

8.9. Proposition. *Let Γ be a monotone operator on $\boldsymbol{X}$, x an element of X and T a tree. Suppose $x \in \mathrm{Ind}(\Gamma, T)$. Then $|x|_\Gamma^T \ll T$.*

Proof. Let τ be a winning strategy of $\exists$ in $G(\Gamma, x, T)$. Let $b \in \sigma\,|x|_\Gamma^T$. Let us play $G(\Gamma, x, T)$ so that $\exists$ follows τ and $\forall$ the ascending chain b of no-losing strategies. We know that $\exists$ cannot lose, so he has to be able to make one more move $f(b) \in T$ after $\forall$ has exhausted b. This f demonstrates $\sigma\,|x|_\Gamma^T \leqslant T$. $\square$

This result is an abstract version of one direction of the equivalence in 4.3 (and 4.8). If we make an additional assumption on Γ, we get also the other direction. Notice that the additional assumption holds in the cases of 4.3 and 4.8.

8.10. Proposition. *Let Γ be a monotone operator on $\boldsymbol{X}$ and assume that for all $x \in X$ there is a smallest set $Y \subseteq X$ with $x \in \Gamma(Y)$. If x is an element of X and T a tree, and if $|x|_\Gamma^T \ll T$, then $x \in \mathrm{Ind}(\Gamma, T)$.*

Proof. Let $f : \sigma\,|x|_\Gamma^T \to T$ be order preserving. We describe the first few moves according to a strategy τ of $\exists$ in $G(\Gamma, x, T)$, the exact definition of τ will then be clear. First $\exists$ plays X_0 and t_0 where X_0 is the smallest set with $x \in \Gamma(X_0)$ and t_0 is the image of the smallest elements s_0 of $\sigma\,|x|_\Gamma^T$ under f. Then $\forall$ picks some $x_1 \in X_0$. The reply of $\exists$ consists of the smallest set X_1 with $x_1 \in \Gamma(X_1)$ and $t_1 = f(s_1)$ where $s_1 \in |x|_\Gamma^T$ is a minimal strategy consistent with picking x_1. It is clear that that $\exists$ can always play according to τ. If $X_\nu \neq \emptyset$ for all ν, then there arises an ω_1-sequence $s_0 \subseteq s_1 \subseteq \cdots$ of elements of $\sigma\,|x|_\Gamma^T$ and $t_0 < t_1 < \cdots$ of elements of T. Hence τ must be a winning strategy of $\exists$. $\square$

9. Stage-comparison

Let Γ and Φ be two monotone operators on $\boldsymbol{X}$ and $\boldsymbol{Y}$ respectively. Let x be an element of X, y an element of Y and T a tree. We shall combine Γ and Φ to get two operators which yield information about the relative sizes of $|x|_\Gamma^T$ and $|y|_\Phi^T$. Our Stage-Comparison Theorem shows that to a certain extent this information is coded in the T-closures and T-coclosures of these operators. We shall then use the Stage-Comparison Theorem in the next section to prove a reduction-type result for ω_1-coinductive relations.

If Γ is monotone on $\boldsymbol{X}$, we define for $A \subseteq \boldsymbol{X}$,

$$\Gamma_0(A) = \Gamma(A) \cup A.$$

9.1. Lemma. $\mathrm{Ind}(\Gamma, T) = \mathrm{Ind}(\Gamma_0, T)$ *and* $\mathrm{Coind}(\Gamma, T) = \mathrm{Coind}(\Gamma_0, T)$. *If* Γ *is positive elementary on a structure, then so is also* Γ_0.

Proof. Assume that $\exists$ has winning strategy τ in $G_0 = G(\Gamma_0, x, T)$. We describe a winning strategy of $\exists$ in $G = G(\Gamma, x, T)$. Denote the moves in G_0 by t'_α, X'_α and x'_α. In G_0, τ gives t'_1 and X'_1, where $x \in \Gamma_0(X'_1)$. If $x \in \Gamma(X'_1)$, then we let $\exists$ play $t_1 = t'_1$ and $X_1 = X'_1$ in G. Otherwise, $x \in X'_1$ and we can let $\forall$ play $x'_1 = x$ in G_0. More generally, we let $\forall$ play $x'_\alpha = x$ in G_0 as long as $x \in X'_\alpha$. Since τ is a winning strategy, there has to be a smallest ordinal α with $x = x'_\alpha \in \Gamma(X'_{\alpha+1})$. Then we are ready to let $\exists$ play $t_1 = t'_{\alpha+1}$ and $X_1 = X'_{\alpha+1}$ in G. Next player $\forall$ plays $x_1 \in X_1$ in G and we do the same for x_1 as what was just done for x. In this way we produce longer and longer initial segments of a play of G_0. It is easy to check that the construction goes over limit steps in G and that it creates a winning strategy for $\exists$. Especially, a limit step in G gives rise to a sequence $(x_\nu)_{\nu<\alpha}$ (α limit) of elements where $x_\nu < x_\mu$ for $\nu < \mu < \alpha$. This corresponds in G_0 to a sequence $(x'_\nu)_{\nu<\alpha'}$ where $x'_\nu \leqslant x'_\mu$ for $\nu < \mu < \alpha'$ and where every x'_ν is x_μ for some μ, and conversely. Hence these sequences have the same limit.

Assume then that $\forall$ has a no-losing strategy σ in G. The following is a no-losing strategy for $\forall$ in G_0. In G_0, $\exists$ plays first t'_1 and X'_1. If $x \in \Gamma(X'_1)$, then we let $\forall$ play x'_1 according to σ. More exactly, we let $\exists$ play $t_1 = t'_1$ and $X_1 = X'_1$ in G. Otherwise $x \in X'_1$ and we let $\forall$ play $x'_1 = x$ in G_0. It is clear that this can be iterated to yield a no-losing strategy for $\forall$ in G_0.

The rest of the Lemma follows immediately from the definitions. $\square$

We use the following two operators to compare the stages connected to two inductive definitions. Let Γ and Φ be monotone on $\boldsymbol{X}$ and $\boldsymbol{Y}$, respectively. Let $\boldsymbol{X} \times \boldsymbol{Y}$ be the cartesian product of X and Y ordered co-ordinatewise. The stage-comparison operators $\Gamma_\leqslant$ and $\Gamma_\ll$ are defined as follows.

$$(x, y) \in \Gamma_\leqslant(B) \Leftrightarrow x \in \Gamma_0(\{x' \mid y \notin \Phi_0(\{y' \mid (x', y') \notin B\})\})$$

and

$$(x, y) \in \Gamma_\ll(B) \Leftrightarrow y \notin \Phi_0(\{y' \mid x \notin \Gamma_0(\{x' \mid (x', y') \in B\})\}).$$

It is straightforward to see that these are monotone. Notice also that if Γ and Φ are positive elementary, then $\Gamma_\leqslant$ and $\Gamma_\ll$ are, too.

These operators are relatively complicated to deal with. Therefore we consider the following two games. The game $G_1(x, y)$ is described in Fig. 3.

So in $G_1(x, y)$ player $\exists$ first chooses an element $t_1 \in T$ and a set X_1 with $x_0 = x \in \Gamma_0(X_1)$ and $x' \geqslant x_0$ for all $x' \in X_1$. Then player $\forall$ chooses an element $x_1 \in X_1$ and a set Y_1 with $y_0 = y \in \Phi_0(Y_1)$ and $y' \geqslant y_0$ for all $y \in Y_1$. The following rounds go in a similar way with the addition that besides e.g. t_2 and X_2 player $\exists$ has also to play an element $y_1 \in Y_1$.

$\exists$	$\forall$	conditions
t_1, X_1		$t_1 \in T, x \in \Gamma_0(X_1)$
	x_1, Y_1	$x_1 \in X_1, y \in \Phi_0(Y_1)$
t_2, y_1, X_2		$t_2 > t_1, y_1 \in Y_1, x_1 \in \Gamma_0(X_2)$
	x_2, Y_2	$x_2 \in X_2, y \in \Phi_0(Y_2)$
$\vdots$	$\vdots$	$\vdots$

Fig. 3. The game $G_1(x, y)$.

For limit α, we first let $x_\alpha = \lim_{\beta<\alpha} x_\beta$ and $y_\alpha = \lim_{\beta<\alpha} y_\beta$ and then $\exists$ chooses t_α with $t_\alpha > t_\beta$ for all $\beta < \alpha$ and X_α with $x_\alpha \in \Gamma_0(X_\alpha)$ and $x' \geqslant x_\alpha$ for all $x' \in X_\alpha$, after which $\forall$ chooses $x_{\alpha+1} \in X_\alpha$ and Y_α with $y_\alpha \in \Phi_0(Y_\alpha)$ and $y' \geqslant y_\alpha$ for all $y' \in Y_\alpha$. Player $\exists$ wins this game if for some α, $x_\alpha \in \Gamma_0(\emptyset)$ and for no $\beta < \alpha$, $y_\beta \in \Phi_0(\emptyset)$. Player $\forall$ wins, if for some α, $y_\alpha \in \Phi_0(\emptyset)$ and for no $\beta \leqslant \alpha$, $x_\beta \in \Gamma_0(\emptyset)$, or if $\exists$ cannot choose t_α.

Let $G_2(x, y)$ be the game described in Fig. 4.

Again for limit α, we first let $x_\alpha = \lim_{\beta<\alpha} x_\beta$ and $y_\alpha = \lim_{\beta<\alpha} y_\beta$. Then $\exists$ chooses $t_\alpha > t_\beta$ for all $\beta < \alpha$ and X_α with $x_\alpha \in \Gamma_0(X_\alpha)$ and $x' \geqslant x_\alpha$ for all $x' \in X_\alpha$, after which $\forall$ chooses Y_α with $y_\alpha \in \Phi_0(Y_\alpha)$ and $y' \geqslant y_\alpha$ for all $y' \in Y_\alpha$. Next $\exists$ chooses $y_\alpha \in Y_\alpha$ and $\forall$ chooses $x_\alpha \in X_\alpha$. Player $\exists$ wins the game if for some α, $x_\alpha \in \Gamma_0(\emptyset)$ and for no $\beta \leqslant \alpha$, $y_\beta \in \Phi_0(\emptyset)$. Player $\forall$ wins if for some α, $y_\alpha \in \Phi_0(\emptyset)$ and for no $\beta < \alpha$, $x_\beta \in \Gamma_0(\emptyset)$, or if $\exists$ cannot choose t_α.

The following lemma is our main tool in getting information about $\Gamma_\leqslant$ and $\Gamma_\ll$. For example, one can show directly from the definitions that $\mathrm{Ind}(\Gamma_\ll, T) \subseteq \mathrm{Ind}(\Gamma_\leqslant, T)$ and $\mathrm{Coind}(\Gamma_\leqslant, T) \subseteq \mathrm{Coind}(\Gamma_\ll, T)$, but the task becomes much easier,

$\exists$	$\forall$	conditions
t_1		$t_1 \in T$
	Y_1	$y \in \Phi_0(Y_1)$
X_1, y_1		$x \in \Gamma_0(X_1), y_1 \in Y_1$
	x_1	$x_1 \in X_1$
t_2		$t_2 > t_1$
	Y_2	$y_2 \in \Phi_0(Y_2)$
X_2, y_2		$x_1 \in \Gamma_0(X_2), y_2 \in Y_2$
$\vdots$	$\vdots$	$\vdots$

Fig. 4. The game $G_2(x, y)$.

if one argues in terms of the games G_1 and G_2. Assume that $\exists$ has a winning strategy τ in $G_2(x, y)$. The following simulation constitutes a winning strategy of $\exists$ in $G_1(x, y)$. For notational simplicity, we denote the subsets of Y played by $\forall$ in $G_2(x, y)$ by Y'_ν instead of Y_ν. At first in $G_2(x, y)$, τ gives $t_1 \in T$ which is part of the first move of $\exists$ in $G_2(x, y)$. Then we let $\forall$ play in $G_2(x, y)$ $Y'_1 = \{y\}$, to which τ gives an answer consisting of X_1 and y_1. This X_1 is the other half of the first move of $\exists$ in $G_2(x, y)$. After this, it is straightforward to read the moves of $\exists$ in $G_2(x, y)$ from those given by τ in $G_1(x, y)$. At limit steps we apply the same trick putting $Y'_{\alpha+1} = \{y_\alpha\}$. The other inclusion is verified in a similar way. Notice that here we really need the definition of the operator Φ_0.

9.2. Lemma. *The games $G(\Gamma_{\leqslant}, (x, y), T)$ and $G_1(x, y)$ are equivalent in the sense that player $\exists$ has a winning strategy in one, if and only if $\exists$ has a winning strategy in the other, and player $\forall$ has a no-losing strategy in one, if and only if $\forall$ has a no-losing strategy in the other. The games $G(\Gamma_{\ll}, (x, y), T)$ and $G_2(x, y)$ are equivalent in a similar way.*

Proof. We prove the four implications concerning $G_1(x, y)$ and then discuss how to prove those concerning $G_2(x, y)$.

(1) Assume first that player $\forall$ has a no-losing strategy τ in $G(\Gamma_{\leqslant}, (x, y), T)$. We describe a no-losing strategy of $\forall$ in $G_1(x, y)$. This is the most difficult part of the proof, and also the most interesting one since it gives a good insight to the role of the operator $\Gamma_{\leqslant}$. Assume that t_1 is the element of T played by $\exists$ on the first round of $G_1(x, y)$. Let B be the set of such pairs (x', y') that there is a set B_1 with $(x, y) \in \Gamma_{\leqslant}(B_1)$ and τ gives (x', y') if the first move of $\exists$ in $G(\Gamma_{\leqslant}, (x, y), T)$ consists of t_1 and B_1. Especially, $x \leqslant x'$ and $y \leqslant y'$ whenever $(x', y') \in B$. It follows that $(x, y) \notin \Gamma_{\leqslant}(-B)$, where $-B$ is the complement of B. This implies by the definition of $\Gamma_{\leqslant}$ that whenever $x \in \Gamma_0(X_1)$, there has to be some $x_1 \in X_1$ with

$$y \in \Phi_0(\{y' \mid (x_1, y') \in B\}).$$

Here $x \leqslant x_1$. Assume that $\exists$ plays in $G_1(x, y)$ on the first round besides t_1 a set X_1. We can assume that $x \in \Gamma_0(X_1)$. Then the first move of $\forall$ is to play $x_1 \in X_1$ as above and the set $Y_1 = \{y' \mid (x_1, y') \in B\}$. On the second round $\exists$ plays in $G_1(x, y)$ among other things an element $y_1 \in Y_1$ with $y \leqslant y_1$. By the definition of Y_1, $(x_1, y_1) \in B$. So the definition of B links (x_1, y_1) to a set B_1 with $(x, y) \in \Gamma(B_1)$. We let $\exists$ play t_1 and B_1 on the first round of $G(\Gamma_{\leqslant}, (x, y), T)$. Then the strategy τ gives exactly the pair (x_1, y_1), and we can repeat the whole argument. It is easy to check that this construction can be carried also over limit steps of the games and that it constitutes a no-losing strategy for $\forall$.

(2) We assume that $\exists$ has a winning strategy τ in $G_1(x, y)$, and describe a winning strategy of $\exists$ in $G(\Gamma_{\leqslant}, (x, y), T)$. On the first round of $G_1(x, y)$, τ gives t_1 and X_1. If $\forall$ would play in $G_1(x, y)$ an element $x_1 \in X_1$ and a set Y_1 where $y \in \Phi_0(Y_1)$, then τ would give a reply consisting of some t_2, y_1, and X_2. This

induces especially a choice function f_{x_1} which maps Y_1 to y_1. So we are able to define B_1 to be the set of all pairs (x', y') where $x' \in X_1$ and $y' = f_{x'}(Y')$ for some set Y' where $y \in \Phi_0(Y')$.

The first move of $\exists$ in $G(\Gamma_{\leqslant}, (x, y), T)$ consists of t_1 and B_1. If in $G(\Gamma_{\leqslant}, (x, y), T)$ $\forall$ replies with the pair (x_1, y_1), then y_1 has to be $f_{x_1}(Y_1)$ for some set Y_1 where $y \in \Phi_0(Y_1)$. To go on, we let $\forall$ play in $G_1(x, y)$ x_1 and Y_1. Then by our choices, τ gives in $G_1(x, y)$ a reply consisting of t_2, y_1 and X_2. Thus we can go on with the construction. It is easy to see that this yields a winning strategy of $\exists$ in $G(\Gamma_{\leqslant}, (x, y), T)$.

(3) We assume that $\forall$ has a no-losing strategy τ in $G_1(x, y)$ and describe a no-losing strategy of $\forall$ in $G(\Gamma_{\leqslant}, (x, y), T)$. Let the first move of $\exists$ in $G(\Gamma_{\leqslant}, (x, y), T)$ consist of t_1 and B_1. We can assume that $x \in \Gamma_0(X_1)$ where $X_1 = \{x' \mid y \notin \Phi_0(\{y' \mid (x', y') \notin B_1\})\}$. We let the first move of $\exists$ in $G_1(x, y)$ consist of t_1 and X_1. To these, τ gives a reply consisting of $x_1 \in X_1$ and a set Y_1 with $y \in \Phi_0(Y_1)$. By the assumption made above, there is some $y_1 \in Y_1$ with $(x_1, y_1) \in B_1$. We let $\forall$ play on the first round of $G(\Gamma_{\leqslant}, (x, y), T)$ the pair (x_1, y_1). This element y_1 is used also as a part of the second move of $\exists$ in $G(\Gamma_{\leqslant}, (x, y), T)$. Otherwise the second and later rounds are exactly as the first one.

Because τ is a no-losing strategy, $X_{\nu+1} \neq \emptyset$ holds for all ν. We can assume that for all ν, $(x_\nu, y_\nu) \in \Gamma_{\leqslant}(B_{\nu+1})$. If $B_{\nu+1} = \emptyset$, then the definition of $\Gamma_{\leqslant}$ implies that $y_\nu \notin \Phi_0(Y)$. Hence $B_{\nu+1} \neq \emptyset$ holds for all ν. So the strategy of $\forall$ described above is no-losing.

(4) Assume that $\exists$ has a winning strategy τ in $G(\Gamma_{\leqslant}, (x, y), T)$. We describe a winning strategy of $\exists$ in $G_1(x, y)$. On the first round of $G(\Gamma_{\leqslant}, (x, y), T)$, τ gives t_1 and B_1. Denote as above $X_1 = \{x' \mid y \notin \Phi_0(\{y' \mid (x', y') \notin B_1\})\}$. Then the first move of $\exists$ in $G_1(x, y)$ consists of t_1 and X_1. The first move of $\forall$ in $G_1(x, y)$ consists of x_1 and Y_1, where $x_1 \in X_1$ and $y \in \Phi_0(Y_1)$. Because τ is a winning strategy, there is some $y' \in Y_1$ with $(x_1, y') \in B_1$. Let y_1 be any of these. Then we can let $\forall$ play in $G(\Gamma_{\leqslant}, (x, y), T)$ the pair (x_1, y_1) to go on in using τ. Again, it is easy to see that this yields a winning strategy for $\exists$ in $G_1(x, y)$.

Finally, we discuss the proof of the assertions concerning $G_2(x, y)$. Assume that $\forall$ has a no-losing strategy τ in $G(\Gamma_{\ll}, (x, y), T)$. In $G_2(x, y)$, $\exists$ plays first $t_1 \in T$. Then as in part (1) above, we consider the set B consisting of all the pairs given by τ as a reply to a move consisting of t_1 and a set B' of $\exists$ in $G(\Gamma_{\ll}, (x, y), T)$. It follows that $(x, y) \notin \Gamma_{\ll}(-B)$. The first move of $\forall$ in $G_2(x, y)$ is the set Y_1 of all those y', where $x \notin \Gamma_0(\{x' \mid (x, y) \notin B\})$. Next $\exists$ plays X_1 and y_1 in $G_2(x, y)$. By the choice of Y_1, there is an element $x_1 \in X_1$ with $(x_1, y_1) \in B$, we let $\forall$ play any x_1 like this. By the definition of B, the pair (x_1, y_1) is a reply given by τ to t_1 and some B_1. We let $\exists$ play in $G(\Gamma_{\ll}, (x, y), T)$ such a B_1 and are able to go on with this construction. The other three implications are verified by direct arguments which are like parts (2)–(4) of the above proof. □

The following theorem approximates the orderings between $|x|_\Gamma^T$ and $|y|_\Phi^T$ from below with the relations $\mathrm{Ind}(\Gamma_{\leqslant}, T)$ and $\mathrm{Ind}(\Gamma_{\ll}, T)$ and from above with the complements of $\mathrm{Coind}(\Gamma_{\leqslant}, T)$ and $\mathrm{Coind}(\Gamma_{\ll}, T)$. Observe below that according to our definition, $\sigma T = T$ when $T = (\omega_1, <)$.

9.3. Stage-Comparison Theorem. *Let Γ and Φ be as above.*

(1) *If $(x, y) \in \mathrm{Ind}(\Gamma_{\leqslant}, T)$, then $x \in \mathrm{Ind}(\Gamma, T)$ and $|x|_\Gamma^T \leqslant |y|_\Phi^T$.*
(2) *If not $|y|_\Phi^T \ll |x|_\Gamma^{\sigma T}$, then $(x, y) \in \mathrm{Ind}'(\Gamma_{\leqslant}, T)$.*
(2′) *If $|x|_\Gamma^T \leqslant |y|_\Phi^T$, then $(x, y) \in \mathrm{Ind}'(\Gamma_{\leqslant}, \sigma T)$.*
(3) *If $(x, y) \in \mathrm{Ind}(\Gamma_{\ll}, T)$, then $x \in \mathrm{Ind}(\Gamma, T)$ and $|x|_\Gamma^T \ll |y|_\Phi^{\sigma T}$.*
(3′) *If $(x, y) \in \mathrm{Ind}(\Gamma_{\ll}, T)$, then $x \in \mathrm{Ind}(\Gamma, T)$ and $|y|_\Phi^T \nleqslant |x|_\Gamma^T$.*
(4) *If $\Phi = \Phi_0$ and $|y|_\Phi^T \nleqslant |x|_\Gamma^T$, then $(x, y) \in \mathrm{Ind}'(\Gamma_{\ll}, T)$.*

Proof of (1). Assume that $(x, y) \in \mathrm{Ind}(\Gamma_{\leqslant}, T)$ and let τ be a winning strategy of $\exists$ in $G_1(x, y)$. We let $\forall$ play in $G_1(x, y)$ so that $Y_\nu = \{y\}$ for all ν. Then $G_1(x, Y)$ becomes essentially $G(\Gamma_0, x, T)$. Thus τ induces a winning strategy τ' of $\exists$ in $G(\Gamma_0, x, T)$. It is then easy to get a winning strategy of $\exists$ in $G(\Gamma, x, T)$ by e.g. playing in $G(\Gamma_0, x, T)$ $x_\nu = x$ as long as $x \in X_\nu$. Then there has to be a smallest ν with $x = x_\nu \in \Gamma(X_{\nu+1})$. In this case, we let $\exists$ begin $G(\Gamma_0, x, T)$ with $X_1' = X_{\nu+1}$ and $t_1' = t_{\nu+1}$.

Consider then the other part of (1). By Lemma 8.1, it suffices to prove that $y \in \mathrm{Coind}(\Phi, |x|_\Gamma^T)$. The following simulation yields a required no-losing strategy for $\forall$ in $H = G(\Phi, y, |x|_\Gamma^T)$. The first move of $\exists$ in H consists of a set Y_1 and an element u_1 of $|x|_\Gamma^T$. Recall that u_1 is a pair whose first co-ordinate is a strategy s_1. Let the first move of $\exists$ according to τ in $G_1(x, y)$ consist of t_1' and X_1'. We can assume that $x \in \Gamma_0(X_1') = X_1' \cup \Gamma(X_1')$. There are two possibilities. In case $x \in \Gamma_0(X_1')$, we use s_1 to get x_1' from t_1' and X_1'. Then we let $\forall$ play in $G_1(x, y)$ $Y_1' = Y_1$. To these τ gives a reply consisting of t_2', y_1' and X_1'. The first move of $\forall$ in H is then $y_1 = y_1'$. The second possibility is that $x \in X_1' - \Gamma(X_1')$. In this case $\forall$ plays in $G_1(x, y)$, $x_1' = x$ and $Y_1' = \{y\}$. Then s_1 is left for later use and we go on with $G_1(x, y)$ before finding out the first move of $\forall$ in $G_1(x, y)$. Notice that in this case, X_1' is not a legal move in $G(\Gamma, x, T)$. Because τ is a winning strategy for $\exists$ in $G_1(x, y)$, the latter possibility cannot occur too often and hence it is easy to see that this simulation can be iterated and that it yields a no-losing strategy for $\forall$ in H. □

Proof of (2). Assume that $(x, y) \in \mathrm{Coind}(\Gamma_{\leqslant}, T)$. It is enough to prove that $|y|_\Phi^T \ll |x|_\Gamma^{\sigma T}$. Let τ be a no-lose strategy of $\forall$ in the game $G_1(x, y)$. We construct an order preserving function $f : \sigma\, |y|_\Phi^T \to |x|_\Gamma^{\sigma T}$ as follows. Recall first that the arguments of f will be initial segments of branches of $|y|_\Phi^T$. The first move of $\exists$ in $G_1(x, y)$ consists of t_1 and X_1 to which τ gives a reply consisting of x_1 and Y_1. We let in $G_1(x, y)$ t_1 and X_1 vary keeping $x \in \Gamma(X_1)$ true, and obtain a function $(t_1, X_1) \to x_1$. The value $f(\emptyset)$ is the pair of this function and $\emptyset$ (empty sequence in T). Let $u_0 \in |y|_\Phi^T$ correspond to a one-move strategy s_0. We shall go on with the game $G_1(x, y)$ to determine the value $f(\langle u_0\rangle)$. Let y_1 be the reply given by s_0 to t_1 and Y_1 in the game $G(\Phi, y, t)$. As above, we let in $G_1(x, y)$ also t_2 and X_2 vary, to these τ gives replies x_2 and Y_2. The functions $(t_1, X_1)\rightarrow x_1$ and $(t_1, X_1, t_2, X_2)\rightarrow x_2$ determine the strategy component of $f(\langle u_0\rangle)$. The corresponding sequence of elements is determined by that in u_0. It is clear that this construction can be iterated and that it gives the required embedding. □

Proof of (2′). Assume that $(x, y)\in \mathrm{Coind}(\Gamma_{\leqslant}, \sigma T)$. We have to show that $|x|^T_\Gamma \not\leqslant |y|^T_\Phi$. By Proposition 8.1, it suffices to show that $x\in \mathrm{Coind}(\Gamma, \sigma\,|y|^T_\Phi)$. Let the game $G_1'(x, y)$ be like $G_1(x, y)$ with the exception that T is replaced by σT. Then ∀ has a no-lose strategy τ in G_1' since $(x, y)\in \mathrm{Coind}(\Gamma_{\leqslant}, \sigma T)$. We shall describe a no-losing strategy of ∀ in the game $H = G(\Gamma, x, \sigma\,|y|^T_\Phi)$. The first move of ∃ in H consists of $u_1\in \sigma\,|y|^T_\Phi$ and X_1. Let t_1 be the root of σT. We let the first move of ∃ in $G_1'(x, y)$ consist of t_1 and X_1. To this, τ gives a reply consisting of $x_1\in X_1$ and Y_1. The first move of ∀ in H is now this x_1. The second move of ∃ in H consists of u_2 and X_2. Here u_2 is a pair of an ascending sequence of elements of T with a last element t_2 and a strategy $s_2\in \sigma\,|y|^T_\Phi$. To t_2 and Y_1 the strategy s_2 gives a reply y_1. We let the second move of ∃ in $G_1'(x, y)$ consist of t_2, y_1 and X_2. To these τ replies with $x_2\in X_2$ and Y_2. The second move of ∀ in H according to the strategy we are describing is then x_2. It is clear that this simulation can be iterated to obtain the desired no-losing strategy. □

Proof of (3). Assume that $(x, y)\in \mathrm{Ind}(\Gamma_{\ll}, T)$ and that τ is a winning strategy of ∃ in $G_2(x, y)$. We let ∀ play in $G_2(x, y)$ so that $Y_\nu = \{y\}$ for all ν. Then $G_2(x, y)$ becomes essentially $G(\Gamma_0, x, T)$ as in the proof of (1) above, and τ induces a winning strategy of ∃ in $G(\Gamma_0, x, T)$. It is easy to use this to get a winning strategy of ∃ in $G(\Gamma, x, T)$.

Then assume that $(x, y)\in \mathrm{Ind}(\Gamma_{\ll}, T)$. Assume by Lemma 9.2 that τ is a winning strategy of ∃ in $G_2(x, y)$. We show that $|x|^T_\Gamma \ll |y|^{\sigma T}_\Phi$, i.e., $\sigma\,|x|^T_\Gamma \leqslant |y|^{\sigma T}_\Phi$. The smallest element of $\sigma\,|x|^T_\Gamma$ is the empty initial segment $\emptyset$ of (every) branch of $|x|^T_\Gamma$. This is mapped to the following element (σ_0, b_0) of $|y|^{\sigma T}_\Phi$. Here b_0 is that chain in σT whose only element is $\emptyset$, the empty initial segment of a branch of T, and σ_0 is the strategy picking from Y_1 such y_1 that τ gives y_1, if ∀ plays Y_1 in $G_2(x, y)$. We consider here only those Y_1 where $y\in \Phi(Y_1)$. Since τ is a winning strategy, σ_0 is no-losing. Consider next an immediate successor of $\emptyset$ in $\sigma\,|x|^T_\Gamma$. It is of the form $\{(s_0, a_0)\}$ where a_0 is a singleton $\{u_0\}$ for some $u_0\in T$ and s_0 is a no-losing strategy of ∀ in $G(\Gamma, x, a_0)$. The image of $\{(s_0, a_0)\}$ will depend only on the answer x_1 given by s_0 to the move u_0, X_1 in $G(\Gamma, x, a_0)$. Here X_1 is the set given above by τ. Given x_1, τ gives first an element $t_2\in T$. Then if we vary Y_2 as Y_1 was varied above, we obtain a function $Y_2\mapsto y_2$. Then we map $\{(s_0, a_0)\}$ to (σ_1, b_1) where $b_1 = \{\emptyset, \{t \mid t\leqslant t_1\}\}$ and where σ_1 is that extension of σ_0 which gives a reply y_2 to Y_2 according to τ. This process can be iterated to give the required embedding. □

Process of (3′). Assume that $(x, y) \in \mathrm{Ind}(\Gamma_{\ll}, T)$. We proved above that $x \in \mathrm{Ind}(\Gamma, T)$. Here we have to show that $|y|^T_\Phi \not\leq |x|^T_\Gamma$. Assume on the contrary that $f: |y|^T_\Phi \to |x|^T_\Gamma$ is order preserving. Let τ be a winning strategy of $\exists$ in $G_2(x, y)$. At first, τ gives an element $t_1 \in T$. Then we vary the set Y_1 and get a function $s_1': Y_1 \mapsto y_1$ where y_1 (and a set X_1) are given by τ as a reply to Y_1. We can consider s_1' as a short strategy s_1 of $\forall$ in $G(\Phi, y, T)$ which is immune relative to the element of T chosen by $\exists$. Then $(s_1, \{t_1\}) \in |y|^T_\Phi$, and $f((s_1, \{t_1\})) \in |x|^T_\Gamma$. Denote $f((s_1, \{t_1\}))$ by (s_1^f, b_1^f). After fixing this notation, we let $\forall$ play $Y_1 = \{y\}$ (and $Y_\alpha = \{y_\alpha\}$, in general). Then $\exists$ plays y_1 and X_1 according to τ. Given X_1, we simulate $G(\Gamma, x, b_1^f)$ and let $\exists$ play there X_1 and the smallest element of b_1^f. To these, s_1^f gives a reply x_1 which we let $\forall$ play in $G_2(x, y)$. It is easy to see that we can go on with this process indefinitely, and that it leads to a play of $G_2(x, y)$ which $\exists$ cannot win. □

Proof of (4). Assume that $(x, y) \in \mathrm{Coind}(\Gamma_{\ll}, T)$ and that τ is a no-losing strategy of $\forall$ in $G_2(x, y)$. We show that $x \in \mathrm{Coind}(\Gamma, |y|^T_\Phi)$ which implies by Lemma 8.5 that $|y|^T_\Phi \leq |x|^T_\Gamma$. The first move of $\forall$ in $G(\Gamma, x, |y|^T_\Phi)$ consists of an element u_1 of $|y|^T_\Phi$ and a set X_1 with $x \in \Gamma(X_1)$. The element u_1 consists of a strategy σ_1 and an initial segment b_1 of a branch of T. Denote by t_1 the last element of b_1 which exists by the definition of $|y|^T_\Phi$. Then the first move of $\exists$ in $G_2(x, y)$ is t_1. Next τ gives a set Y_1 with $y \in \Phi_0(Y_1) = \Phi(Y_1)$. We can apply σ_1 to get y_1. The next move of $\exists$ in $G_2(x, y)$ consists of X_1 and y_1, to which τ gives a reply x_1. This x_1 is the first move of $\forall$ according to the strategy we are describing. It is easy to see that this gives the required no-losing strategy of $\exists$. □

The proof of the Stage-Comparison Theorem is now complete. □

As the remark before the Stage-Comparison Theorem shows, the theorem has a simpler form in the case of ω_1-induction. This special case will be discussed in the next section.

It is well known that the levels of an elementary inductive definition are hyperelementary. We have the following version of this fact. It combines Stage-Comparison of a monotone operator Γ on X and the operator Γ_s of Example 4.2. In part (2) of the following theorem, recall that $T = \sigma T$ if $T = (\omega_1, <)$.

9.4. Theorem. *Assume that T is a tree in which every element of limit height is uniquely determined by its predecessors. If $T \ll U$, then*

(1) $\quad x \in \mathrm{Ind}(\Gamma, T) \Leftrightarrow (x, T) \in \mathrm{Ind}(\Gamma_{\leq}, U) \Leftrightarrow (T, x) \in \mathrm{Coind}(\Gamma_{\ll}, U),$

(2) $\quad x \in \mathrm{Coind}(\Gamma, \sigma T) \Rightarrow (x, T) \in \mathrm{Coind}(\Gamma_{\leq}, U)$

$\quad\quad \Leftrightarrow (T, x) \in \mathrm{Ind}(\Gamma_{\ll}, U) \Rightarrow x \in \mathrm{Coind}(\Gamma, T).$

Proof. Consider first the proof of (1). Below the games $G_1(x, T)$ and $G_2(T, x)$ are modified so that their length is U, i.e., player $\exists$ picks elements from U in an ascending way on every round.

Claim 1. $x \in \mathrm{Ind}(\Gamma, T) \Rightarrow (x, T) \in \mathrm{Ind}(\Gamma_{\leqslant}, U)$.

Proof. Let $f: T \to U$ be order preserving. We may assume that f maps branches of T to initial segments of branches of U. Assume that τ is a winning strategy for $\exists$ in $G(\Gamma, x, T)$. It can be assumed that the heights of the elements of T given by τ form an initial segment of the ordinals, i.e., that $\exists$ chooses always elements from T as low as possible. We shall describe a winning strategy of $\exists$ in $G_1(x, T)$. Here the game $G_1(x, T)$ is defined in terms of the operators Γ and Γ_s, and the length of the game is U. Let the first move of $\exists$ in $G(\Gamma, x, T)$ according to τ consist of t_1 and X_1. By our assumption, t_1 is a minimal element in T. We let the first move of $\exists$ in $G_1(x, T)$ consist of $f(t_1)$ and X_1. Then let x_1 and Y_1 form the first move of $\forall$ in $G_1(x, T)$. Assume that $T \in \Gamma_s(Y_1)$ and not just $T \in (\Gamma_s)_0(Y_1)$. It follows from the definition of Γ_s that Y_1 contains all the subtrees determined by the immediate successors of t_1 in T. We interpret x_1 as the first move of $\forall$ in $G(\Gamma, x, T)$. Let the second move of $\exists$ in $G(\Gamma, x, T)$ consist of t_2 and X_2. Then we let the second move of $\exists$ in $G_1(x, T)$ consist of $f(t_2)$, y_1, X_2 where y_1 is the element of Y_1 to which t_2 belongs. If $T \in Y_1 - \Gamma_s(Y_1)$, then we let $\exists$ play $y_1 = T$. If $T = y_\alpha \in \Gamma_s(Y_{\alpha+1})$, then $y_{\alpha+1}$ is determined by t_2 as above, i.e., t_2 is the root of $y_{\alpha+1}$. Clearly this process can be iterated to yield the required winning strategy. □

Claim 2. $(x, T) \in \mathrm{Ind}(\Gamma_{\leqslant}, U) \Rightarrow (T, x) \in \mathrm{Coind}(\Gamma_{\ll}, U)$.

Let τ be a winning strategy of $\exists$ in $G_1(x, T)$. The first move of $\exists$ in $G_2(T, x)$ is some $v_1 \in U$ which (as well as the other elements v_i) plays no role in the construction of the no-losing strategy of $\forall$ in $G_2(T, x)$. The first move of $\exists$ in $G_1(x, T)$ according to τ consists of $u_1 \in U$ and X_1. The first move of $\forall$ in $G_2(T, x)$ will now be X_1. Then in $G_2(T, x)$, $\exists$ plays x_1 and Y_1 which are used as the first move of $\forall$ in $G_1(x, T)$. To this τ gives in $G_1(x, T)$ u_2, y_1 and X_2. The next two moves of $\forall$ in $G_2(T, x)$ will be these y_1 and X_2, between which $\exists$ picks $v_2 \in U$. It is easy to see that this process gives the required no-losing strategy of $\forall$ in $G_2(T, x)$. Notice that the elements u_i have here only an indirect role: τ picks them in such a way that $\exists$ has enough time to win $G_1(x, T)$. □

Claim 3. $(T, x) \in \mathrm{Coind}(\Gamma_{\ll}, U) \Rightarrow x \in \mathrm{Ind}(\Gamma, T)$.

Proof. Let $f: \sigma T \to U$ be order preserving. Let τ be a no-losing strategy of $\forall$ in $G_2(T, x)$. We describe a winning strategy on $\exists$ in $G(\Gamma, x, T)$. We let $\exists$ play first in $G_2(T, x)$ the element $v_1 = f(\emptyset)$ (where $\emptyset$ is the smallest element of σT). Then τ

gives a set X_1 with $x \in \Gamma_0(X_1) = X_1 \cup \Gamma(X_1)$. The first move of $\exists$ in $G(\Gamma, x, T)$ consists now of t_1 and X_1 where t_1 is the unique root of T. Then $\forall$ plays x_1 in $G(\Gamma, x, T)$ and x_1 is going to be part of the second move of $\exists$ in $G_2(T, x)$. The part Y_1 is the set of subtrees determined by immediate successors of t_1. Then τ gives y_1 and after $\exists$ has played in $G_2(T, x)$ $v_2 = f(\{t_1\})$, also X_2. The second move of $\exists$ in $G(\Gamma, x, T)$ consist of t_2 and X_2 where t_2 is the root of y_1 (and hence an immediate successor of t_1.) This process seems to go on indefinitely, but it is cut down by the fact that T has no uncountable branches. It is easy to see that the strategy described here is a winning strategy for $\exists$ in $G(\Gamma, x, T)$. □

We proceed next to part (2) of the theorem. We prove here only the first and last implication, the middle equivalence is left to the reader.

Claim 4. $x \in \mathrm{Coind}(\Gamma, \sigma T) \Rightarrow (x, T) \in \mathrm{Coind}(\Gamma_{\leqslant}, U)$.

Proof. Assume that τ is a no-losing strategy of $\forall$ in $G(\Gamma, x, \sigma T)$. The following is a no-losing strategy of $\forall$ in $G_1(x, T)$. In $G_1(x, T)$, player $\exists$ gives first u_1 and X_1. We let $\exists$ play in $G(\Gamma, x, \sigma T)$ first $v_1 = \emptyset$, the smallest element of σT, and X_1. To these, τ gives in $G(\Gamma, x, \sigma T)$ a reply x_1. Then we let the first move of $\forall$ in $G_1(x, T)$ consist of x_1 and Y_1, where Y_1 is the set of the subtrees of T of the form $\{t \mid t \geqslant t'\}$ where t' is an immediate successor of the root of T. The next move of $\exists$ in $G_1(x, T)$ consists of u_2, X_2 and y_1. Here $y_1 \in Y_1$ and we can let $\exists$ play in $G(\Gamma, x, \sigma T)$ v_2 and X_2 where v_2 is the root of y_1. It is easy to see that this constitutes the required no-losing strategy of $\forall$ in $G_1(x, T)$. □

Claim 5. $(T, x) \in \mathrm{Ind}(\Gamma_{\ll}, U) \Rightarrow x \in \mathrm{Coind}(\Gamma, T)$.

Proof. Let τ be a winning strategy of $\exists$ in $G_2(T, x)$. The following is a no-losing strategy of $\forall$ in $G(\Gamma, x, T)$. In $G(\Gamma, x, T)$, $\exists$ plays first t_1 and X_1. In $G_2(T, x)$, τ gives at first $u_1 \in U$ (which has no role in this construction.) We let $\forall$ play X_1 in $G_2(T, x)$, to which τ gives x_1 and Y_1. This x_1 is used as the first move of $\forall$ in $G(\Gamma, x, T)$. Next $\exists$ plays t_2 and X_2 in $G(\Gamma, x, T)$. To proceed, we let $\forall$ play in $G_2(T, x)$ y_1 so that $t_2 \in y_1$, and in addition, if $T \in \Gamma_s(Y_1)$, we let y_1 be a subtree of T determined by an immediate successor of the root of T. Otherwise, $T \in Y_1$ and we set $y_1 = T$. After this, τ gives $u_2 \in U$ and we let $\forall$ reply on $G_2(T, x)$ with X_2. Then τ gives x_2 and Y_2 and we can let $\forall$ play x_2 next in $G(\Gamma, x, T)$. This simulation process gives clearly the required no-losing strategy. □

This completes the proof of Theorem 9.4. □

By combining the Stage-Comparison Theorem 9.4, we obtain the following result.

9.5. Corollary. *Assume that T is a tree in which every element of limit height is uniquely determined by its predecessors. If $T \ll U$, then*

$$x \in \operatorname{Ind}(\Gamma, T) \;\Rightarrow\; |x|^U_\Gamma \leqslant T \;\Rightarrow\; x \in \operatorname{Ind}'(\Gamma, \sigma T).$$

Proof. Since $T \leqslant |x|^T_{\Gamma_s} \leqslant T$,

$$\begin{aligned} x \in \operatorname{Ind}(\Gamma, T) &\Rightarrow (x, T) \in \operatorname{Ind}(\Gamma_{\leqslant}, U) \\ &\Rightarrow |x|^U_\Gamma \leqslant |T|^U_{\Gamma_s} \\ &\Rightarrow |x|^U_\Gamma \leqslant T \\ &\Rightarrow (x, T) \in \operatorname{Ind}'(\Gamma_{\leqslant}, \sigma U) \\ &\Rightarrow x \in \operatorname{Ind}'(\Gamma, \sigma T). \qquad \square \end{aligned}$$

10. A reduction theorem for ω_1-induction

Our Stage-Comparison Theorem obtains an especially appealing form in the case of α-induction. If Γ and Φ are as before, then (see Fig. 5):

(1) if $(x, y) \in \operatorname{Ind}(\Gamma_{\leqslant}, \omega_1)$, then $x \in \operatorname{Ind}'(\Gamma, \omega_1)$ and $|x|^{\omega_1}_\Gamma \leqslant |y|^{\omega_1}_\Phi$,
(2) if not $|y|^{\omega_1}_\Phi \ll |x|^{\omega_1}_\Gamma$, then $(x, y) \in \operatorname{Ind}'(\Gamma_{\leqslant}, \omega_1)$,
(3) if $(x, y) \in \operatorname{Ind}(\Gamma_{\ll}, \omega_1)$, then $x \in \operatorname{Ind}'(\Gamma, \omega_1)$ and $|x|^{\omega_1}_\Gamma \ll |y|^{\omega_1}_\Phi$,
(4) if $\Phi_0 = \Phi$ and $|y|^{\omega_1}_\Phi \not\leqslant |x|^{\omega_1}_\Gamma$, then $(x, y) \in \operatorname{Ind}'(\Gamma_{\ll}, \omega_1)$.

This result is strong enough to yield the following weak version of the reduction principle for complements of α-coinductive relations. The proof relies on the Combination and Transitivity Lemmas of Section 6.

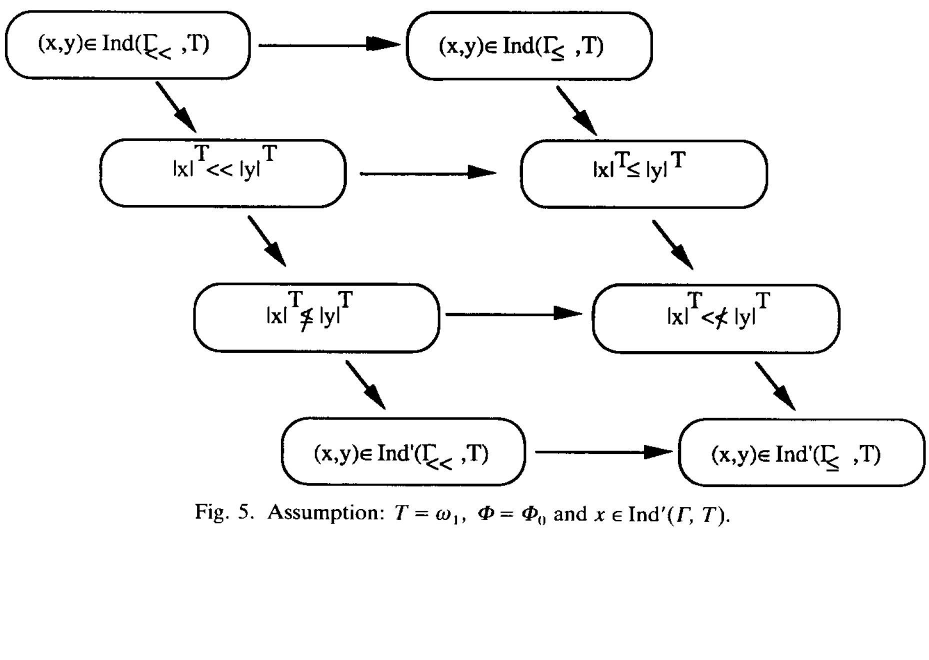

Fig. 5. Assumption: $T = \omega_1$, $\Phi = \Phi_0$ and $x \in \operatorname{Ind}'(\Gamma, T)$.

10.1. Weak Reduction Theorem. *Suppose $\mathfrak{A}$ is a structure around* **X**. *Let P and Q be complements of ω_1-coinductive relations on $\mathfrak{A}$. Then there are complements P_1 and Q_1 of ω_1-coinductive sets such that $P_1 \subseteq P$, $Q_1 \subseteq Q$ and $P \cup Q = P_1 \cup Q_1$. Moreover, there are positive elementary operators Γ_1 and Φ_1 on $\mathfrak{A}$ and parameters $a_1, \ldots, a_m$ and $b_1, \ldots, b_k$ such that*

(1) $\quad P_1(x_1, \ldots, x_n) \Leftrightarrow (x_1, \ldots, x_n, a_1, \ldots, a_m) \in \mathrm{Ind}'(\Gamma_1, \omega_1)$,

(2) $\quad Q_1(x_1, \ldots, x_n) \Leftrightarrow (x_1, \ldots, x_n, b_1, \ldots, b_k) \in \mathrm{Ind}'(\Phi_1, \omega_1)$,

(3) $\quad$ *for no* $x_1, \ldots, x_n$, $\quad (x_1, \ldots, x_n, a_1, \ldots, a_m) \in \mathrm{Ind}(\Gamma_1, \omega_1)$ *and*

$$(x_1, \ldots, x_n, b_1, \ldots, b_k) \in \mathrm{Ind}(\Phi_1, \omega_1).$$

Proof. Let Γ and Φ be positive elementary operators and $a_1, \ldots, a_m$ and $b_1, \ldots, b_k$ be sequences of elements of $\mathfrak{A}$ for which

$$P(x_1, \ldots, x_n) \Leftrightarrow (x_1, \ldots, x_n, a_1, \ldots, a_m) \in \mathrm{Ind}'(\Gamma, \omega_1),$$

$$Q(x_1, \ldots, x_n) \Leftrightarrow (x_1, \ldots, x_n, b_1, \ldots, b_k) \in \mathrm{Ind}'(\Phi, \omega_1).$$

We may assume that $\Gamma = \Gamma_0$, $\Phi = \Phi_0$ and that $(a_1, \ldots, a_m) = (b_1, \ldots, b_k)$. We shall write $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{a} = (a_1, \ldots, a_m)$. Let c and d be two distinct elements of $\mathfrak{A}$. Define

$$R(y, \boldsymbol{x}) \Leftrightarrow [y = c \wedge P(\boldsymbol{x})] \vee [y = d \wedge Q(\boldsymbol{x})].$$

It is easy to show that there is a positive elementary operator Ξ for which

$$R(y, \boldsymbol{x}) \Leftrightarrow (y, \boldsymbol{x}, \boldsymbol{a}) \in \mathrm{Ind}'(\Xi, \omega_1).$$

Consider the operators $\Gamma_{\leqslant}$ and $\Gamma_{\ll}$ comparing Ξ relative to itself and define

$$P_1(\boldsymbol{x}) \Leftrightarrow ((c, \boldsymbol{x}, \boldsymbol{a}), (d, \boldsymbol{x}, \boldsymbol{a})) \in \mathrm{Ind}'(\Gamma_{\leqslant}, \omega_1),$$

$$Q_1(\boldsymbol{x}) \Leftrightarrow ((d, \boldsymbol{x}, \boldsymbol{a}), (c, \boldsymbol{x}, \boldsymbol{a})) \in \mathrm{Ind}'(\Gamma_{\ll}, \omega_1).$$

Again it is easy to see that there are positive elementary operators Γ_1 and Φ_1 for which

$$P_1(\boldsymbol{x}) \Leftrightarrow (c, d, \boldsymbol{x}, \boldsymbol{a}) \in \mathrm{Ind}'(\Gamma_1, \omega_1),$$

$$Q_1(\boldsymbol{x}) \Leftrightarrow (c, d, \boldsymbol{x}, \boldsymbol{a}) \in \mathrm{Ind}'(\Phi_1, \omega_1).$$

Claim 1. $P_1 \subseteq P$.

Proof. If not $P(\boldsymbol{x})$, then $\forall$ has a no-losing strategy in the game $G(\Gamma, (\boldsymbol{x}, \boldsymbol{a}), \omega_1)$. This would induce a no-losing strategy of $\forall$ in the game $G(\Xi, (c, \boldsymbol{x}, \boldsymbol{a}), \omega_1)$ and furthermore one in $G(\Gamma_{\leqslant}, ((c, \boldsymbol{x}, \boldsymbol{a}), (d, \boldsymbol{x}, \boldsymbol{a})), \omega_1)$, implying that not $P_1(\boldsymbol{x})$. $\square$

Claim 2. $Q_1 \subseteq Q$.

This can be shown in the same way as Claim 1.

Claim 3. $P_1 \cup Q_1 = P \cup Q$.

Proof. Let $\boldsymbol{x} \in P \cup Q$. Either

(i) $|(c, \boldsymbol{x}, \boldsymbol{a})|^{\omega_1}_{\Xi} \leqslant |(d, \boldsymbol{x}, \boldsymbol{a})|^{\omega_1}_{\Xi}$, or

(ii) $|(c, \boldsymbol{x}, \boldsymbol{a})|^{\omega_1}_{\Xi} \not\leqslant |(d, \boldsymbol{x}, \boldsymbol{a})|^{\omega_1}_{\Xi}$.

If $P(\boldsymbol{x})$, then $(c, \boldsymbol{x}, \boldsymbol{a}) \notin \mathrm{Coind}(\Xi, \alpha)$ and so in case (i) the Stage-Comparison Theorem implies that $P_1(\boldsymbol{x})$. On the other hand in case (ii), the Stage-Comparison Theorem implies that $Q_1(\boldsymbol{x})$. If not $P(\boldsymbol{x})$ but $Q(\boldsymbol{x})$, then the tree $|(c, \boldsymbol{x}, \boldsymbol{a})|^{\omega_1}_{\Xi}$ contains an ω_1-branch, but $|(d, \boldsymbol{x}, \boldsymbol{a})|^{\omega_1}_{\Xi}$ does not contain one. Again case (ii) and $Q_1(\boldsymbol{x})$ hold. $\square$

Claim 4. *There is no* $\boldsymbol{x}$ *for which* $(c, d, \boldsymbol{x}, \boldsymbol{a}) \in \mathrm{Ind}(\Gamma_1, \omega_1) \cap \mathrm{Ind}(\Phi_1, \omega_1)$.

Indeed, if $(c, d, \boldsymbol{x}, \boldsymbol{a}) \in \mathrm{Ind}(\Gamma_1, \omega_1) \cap \mathrm{Ind}(\Phi_1, \omega_1)$, then the Stage-Comparison Theorem implies that both (i) and (ii) hold above. $\square$

Remark. Complements of ω-coinductive relations are exactly the ω-inductive relations. This is not true of ω_1-inductive relations. Therefore we have to formulate reduction in a roundabout way for complements of ω_1-coinductive relations rather than for ω_1-inductive relations. We suspect that reduction does not hold for ω_1-inductive relations. If $V = L$, then reduction fails for $\boldsymbol{\Pi}^2_1$ (see e.g. [2, p. 341]), so in this case by Example 4.4, we cannot improve the above Reduction Theorem to $P_1 \cap Q_1 = \emptyset$.

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